Lecture 4: Euler's Formula and Applications

Theorem - $e^{i\theta} = \cos\theta + i\sin\theta$

"proof"

- $e^{z} = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \ldots + \frac{1}{n!}z^n + \ldots$
- $e^{i\theta} = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \ldots + \frac{1}{n!}(i\theta)^n + \ldots$

$$= 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \ldots$$

$$= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \ldots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots\right)$$

- $\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \ldots$
- $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots$

Therefore,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Definition - If $z \in \mathbb{C}$, then

$$e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos(y) + i\sin(y))$$

Consequences:

1. $e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$
2. $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

Polar $\rightarrow$ Exponential Form

$z = x + iy$

$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = re^{i\theta} = |z|e^{i\theta}$

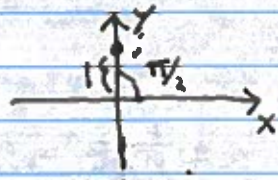
$\Rightarrow$ Three forms of a complex number:

$z = x + iy$, $z = r(\cos\theta + i\sin\theta)$, $z = re^{i\theta}$

Standard form — Polar form — Exponential form

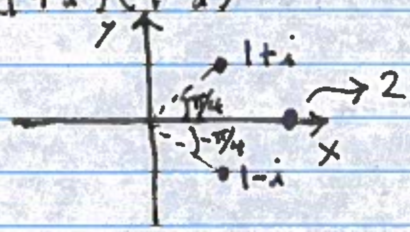
Examples

1. i^i



$$i^i = (e^{i\pi/2})^i = e^{-\pi/2}.$$

2. $(1+i)(1-i)$



$$1+i = \sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4)) = \sqrt{2}\,e^{i\pi/4}$$
$$1-i = \sqrt{2}\,(\cos(-\pi/4) + i\sin(-\pi/4)) = \sqrt{2}\,e^{-i\pi/4}$$

$$\Longrightarrow (1+i)(1-i) = (\sqrt{2}\,e^{i\pi/4})(\sqrt{2}\,e^{-i\pi/4}) = 2\,e^{0} = 2$$

* Complex multiplication:
- multiply magnitudes and add angles!

3. $$\frac{1+i}{1-i} = \frac{\sqrt{2}\,e^{i\pi/4}}{\sqrt{2}\,e^{-i\pi/4}} = e^{i\pi/2} = i.$$

* Complex division:
- divide magnitudes and subtract angles!

4. $$\begin{aligned}(1+i)^{101} = (\sqrt{2}\,e^{i\pi/4})^{101} &= 2^{101/2}\,e^{101\pi/4} \\ &= 2^{101/2}\,e^{100\pi/4}\,e^{i\pi/4} \\ &= 2^{101/2}\,e^{25\pi i}\,e^{\pi i/4} \\ &= 2^{101/2}(-1)(1/\sqrt{2} + i/\sqrt{2}) \\ &= (-1)\,2^{50}(1+i).\end{aligned}$$

5. Let $n \in \mathbb{N}$, then
$$z^n = (|z|e^{i\theta})^n = |z|^n e^{in\theta} = |z|^n(\cos(n\theta) + i\sin(n\theta)).$$

6. $z = |z|e^{i\theta} = |z|e^{i\theta + 2n\pi}, \quad n \in \mathbb{Z}.$

Trigonometry

1. $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

2. $\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$

   proof:

   - $e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$
   - $e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

   $= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$.

   $\Rightarrow \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

   $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

3. $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$.

   proof:

   - $(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$
   - $(e^{i\theta})^2 = e^{2i\theta} = \cos(2\theta) + i\sin 2\theta$

   $\Rightarrow \cos^2\theta - \sin^2\theta = \cos 2\theta$, $2\cos\theta\sin\theta = \sin 2\theta$

   $\Rightarrow \cos^2\theta - 1 + \cos^2\theta = \cos 2\theta$, $\sin 2\theta = 2\cos\theta\sin\theta$

   $\Rightarrow 2\cos^2\theta = \cos 2\theta + 1$, $\sin 2\theta = 2\cos\theta\sin\theta$

   $\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$