Lecture 5: Powers and Roots

Roots of Unity:

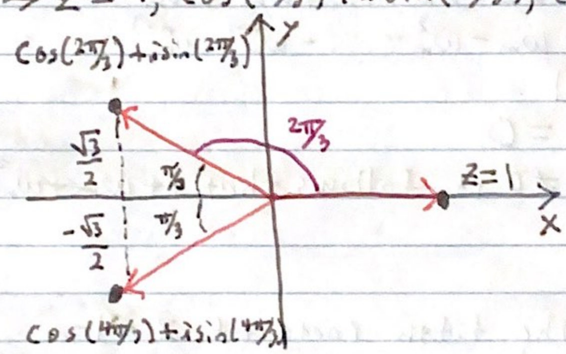
1. Solve the equation

$$z^3 = 1$$

Let $1 = 1 \cdot e^{2\pi i n}$, $n \in \mathbb{Z}$.

$\Rightarrow z = e^{2\pi/3\, i n}$, $n \in \mathbb{Z}$

$\Rightarrow z = 1, \cos(2\pi/3) + i\sin(2\pi/3), \cos(4\pi/3) + i\sin(4\pi/3)$



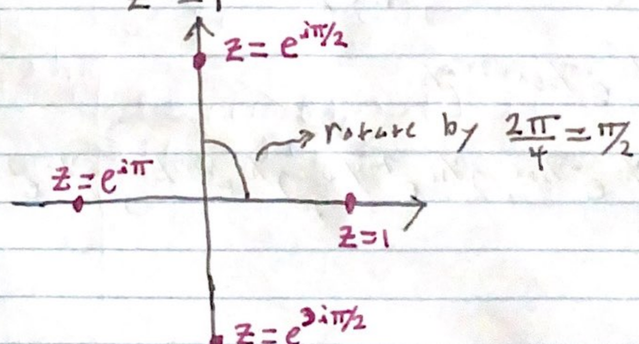
$\Rightarrow z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i \longrightarrow$ Three cube roots of unity

$z = 1, w_3, w_3^2$

indicates first nontrivial cube roots of unity.

2. Solve the equation

$$z^4 = 1$$



$w_4 = e^{i\pi/2}$

$z = 1, w_4, w_4^2, w_4^3$

$= 1, e^{i\pi/2}, e^{i\pi}, e^{3i\pi/2}$.

3. Solve $z^5 = 1$

$\Rightarrow w_5 = e^{2\pi i/5}$

$\Rightarrow z = 1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$

Theorem - $1 + w_m + w_m^2 + \dots + w_m^{m-1} = 0$

~~proof~~:

$$(w_m - 1)(1 + w_m + w_m^2 + \dots + w_m^{m-1})$$
$$= w_m + w_m^2 + w_m^3 + \dots + w_m^{m-1} + w_m^m$$
$$-1 - w_m - w_m^2 - \dots - w_m^{m-2} - w_m^{m-1}$$
$$= w_m^m - 1$$
$$= 1 - 1 = 0.$$

Since $w_m - 1 \neq 0$ it follows that $1 + w_m + w_m^2 + \dots + w_m^{m-1} = 0$.

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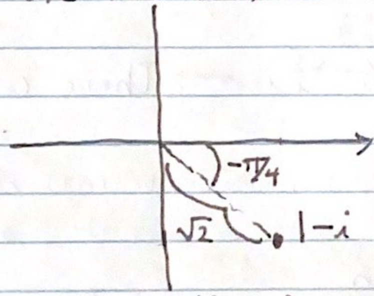
Example:

What are the fifth roots of $z_0 = 1 - i$.



$$1 - i = \sqrt{2}\, e^{-\pi/4 i + 2\pi i n}, \quad n \in \mathbb{Z}$$

$$\Rightarrow (1-i)^{1/5} = 2^{1/10} e^{-\pi/20 i + 2\pi i n/5}$$

$$\Rightarrow (1-i)^{1/5} = 2^{1/10} e^{-\pi/20 i}, 2^{1/10} e^{7\pi i/20}, 2^{1/10} e^{15\pi i/20}, 2^{1/10} e^{23\pi i/20}, e^{31\pi i/20}.$$

Example:

Solve the equation

$$(z+1)^3 = z^3$$

$$\Rightarrow \left(\frac{z}{z+1}\right)^3 = 1$$

$$\Rightarrow \frac{z}{z+1} = 1,\ e^{2\pi i/3},\ e^{4\pi i/3}$$

$$\Rightarrow z = z+1,\quad z = e^{2\pi i/3} z + e^{2\pi i/3},\quad z = e^{4\pi i/3} z + e^{4\pi i/3}$$

$\downarrow$ no solution

$$\Rightarrow z = \frac{e^{2\pi i/3}}{1 - e^{2\pi i/3}}$$

$$z = \frac{e^{4\pi i/3}}{1 - e^{4\pi i/3}}$$