Lecture 8: Functions of a Complex Variable:

Representation of Functions

$f: \mathbb{C} \to \mathbb{C}$ defined by:

$$f(z) = z^2$$

Other ways to represent functions

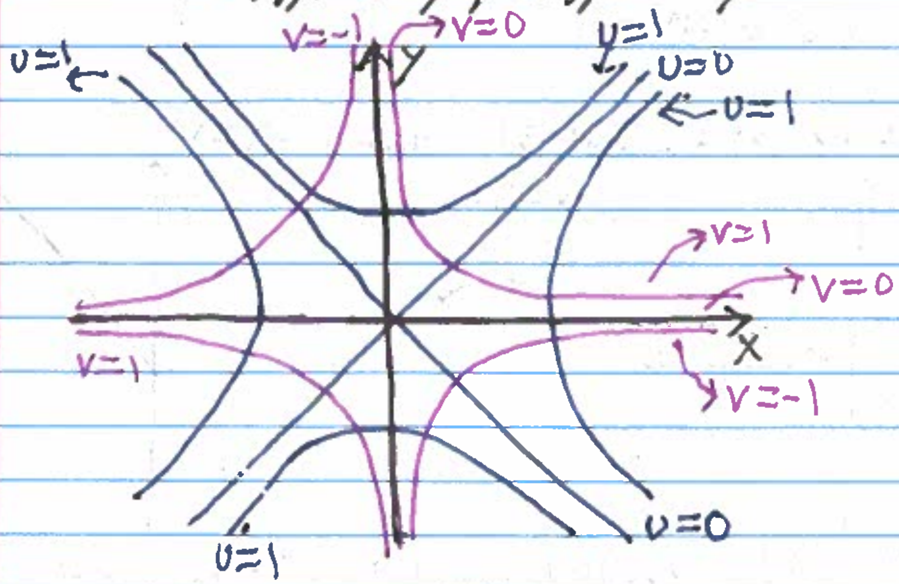
$$W = f(z) = u(z) + i v(z) = u(x,y) + i v(x,y) \quad \text{(Cartesian)}$$

$$W = f(z) = \rho(r,\theta)\, e^{i\phi(r,\theta)} \quad \text{(polar)}$$

Example:

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$\Rightarrow u(x,y) = x^2 - y^2,\ v(x,y) = 2xy$$



*Conjecture: Contours meet at right angles:

$$\nabla u = \langle 2x, -2y \rangle$$

$$\nabla v = \langle 2y, 2x \rangle$$

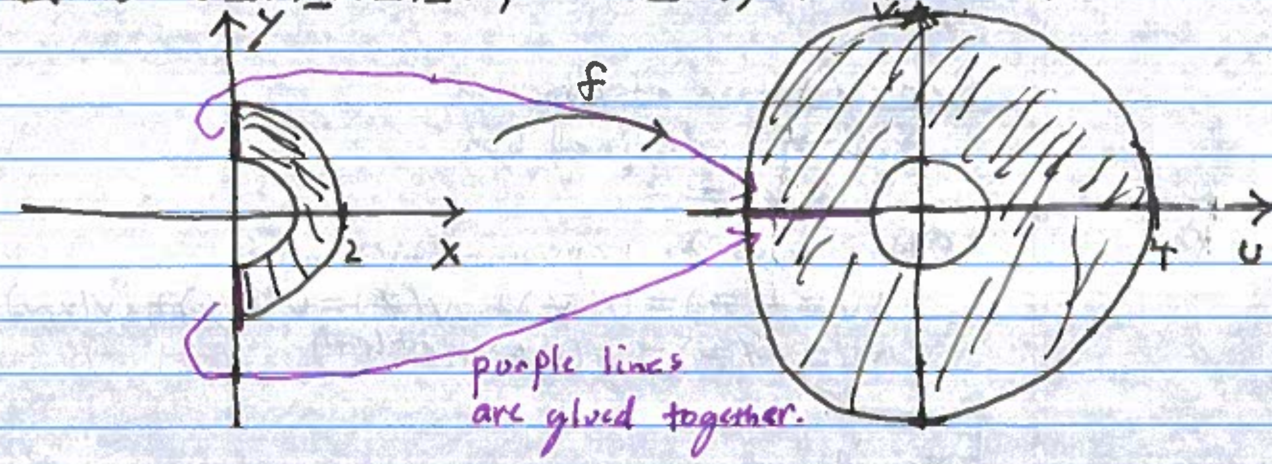
$$\Rightarrow \nabla u \cdot \nabla v = 4xy - 4xy = 0!!$$

In polar form:

$$f(z) = (re^{i\theta})^2 = r^2 e^{2i\theta}$$

$$\Rightarrow \rho(r,\theta) = r^2,\ \phi(r,\theta) = 2\theta.$$

If $D = \{z : 1 \leq |z| \leq 2,\ Re(z) \geq 0\}$, what is $f(D)$?

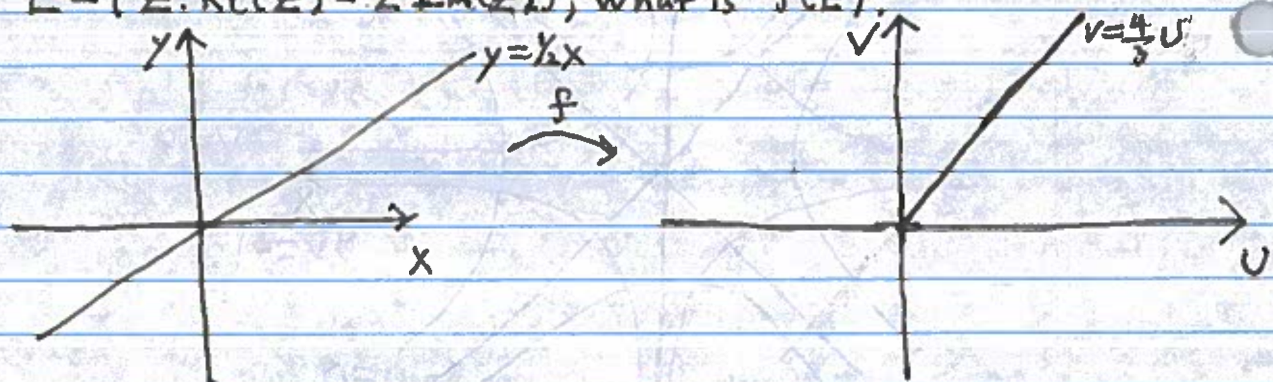


$f : D \to \mathbb{R}^2$ is not one to one since

$f(i) = i^2 = -1$

$f(-i) = (-i)^2 = -1$

$E = \{z : Re(z) = 2Im(z)\}$, what is $f(E)$?



$f(z)|_E = f(2y + iy) = (2y+iy)^2 = 4y^2 + 4iy^2 - y^2 = 3y^2 + 4iy^3$

$\Rightarrow u + iv = 3y^2 + 4iy^2$

$\Rightarrow v = \frac{4}{3}u, \quad u, v > 0.$

$G = \{z : m\,Re(z) = Im(z)\}$, where $m \in \mathbb{R}$

$f(z)|_G = f(x + imx) = x^2 - m^2x^2 + 2imx^2$

$= x^2(1-m^2) + 2imx^2$

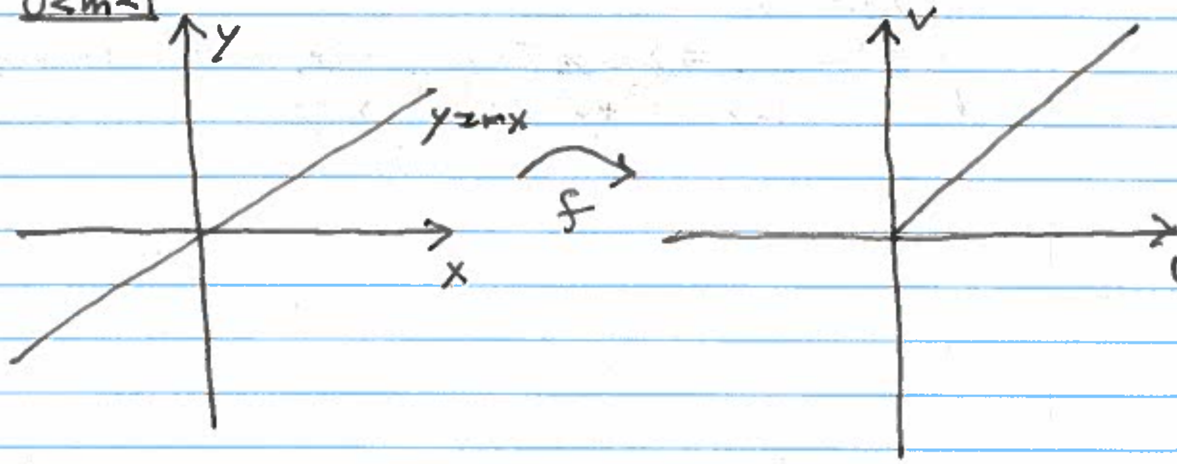
Let $t = x^2$

$\Rightarrow u(t) = (1-m^2)t^2$

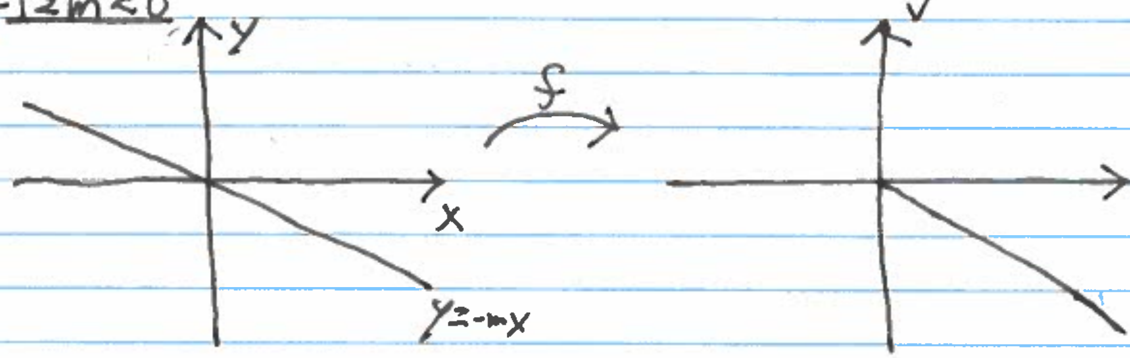
$v(t) = 2mt^2$

4 possibilities:

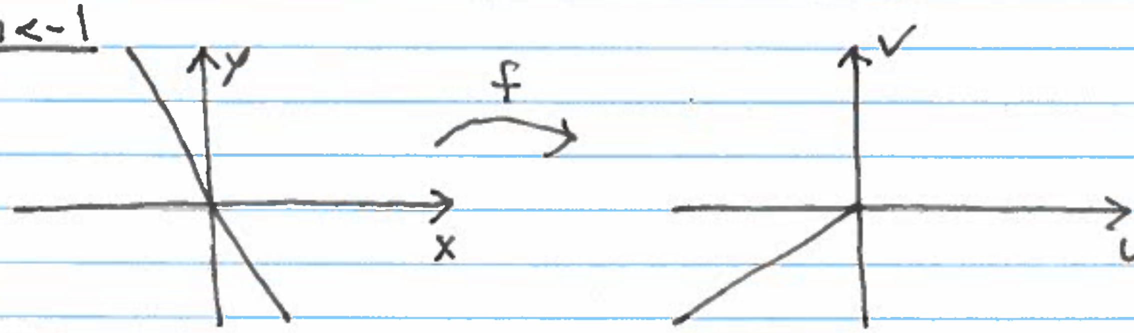
$0<m<1$



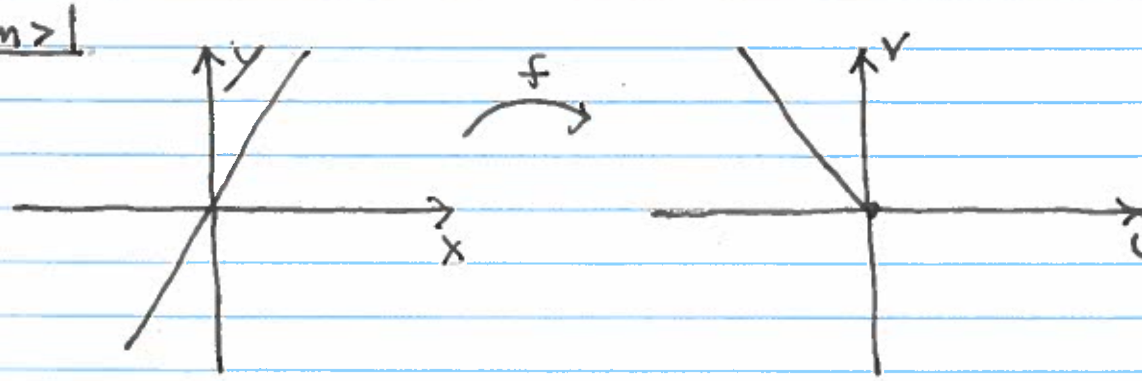
$-1<m<0$



$m<-1$



$m>1$



Example:

$$g(z) = 1/z$$

$$g(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = u(x,y) + iv(x,y)$$