MTH 351/651
Fall 2022
Exam 1
09/23/22

Name (Print): Key

This exam contains 8 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	5	
3	5	
4	15	
5	15	
6	10	
7	25	
8	15	
Total:	100	

1. (10 points) (**True or False**) Identify whether the following statements are true or false. If a statement is false provide a counterexample or *briefly* explain why you think the statement is false. If a statement is true simply write true, no explanation is necessary.

   (a) (2.5 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$. If $x(t)$ is a solution to this dynamical system then $\ddot{x} = f'(x)$.

   False, $\ddot{x} = f(x)f'(x)$

   (b) (2.5 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be smooth and consider the dynamical system $\dot{x} = f(x)$. If this system has two fixed points then these fixed points can both be stable.

   False, there must be a fixed point in between.

   (c) (2.5 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function. The point x^* is an unstable fixed for this equation $\dot{x} = f(x)$ if and only if $f'(x^*) > 0$.

   False,
   $\dot{x} = x^3$
   has $x = 0$ as an unstable fixed point.

   (d) (2.5 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the differential equation $\dot{x} = f(x)$ which has one fixed point x^* which is unstable. If $x(t)$ solves this differential equation then

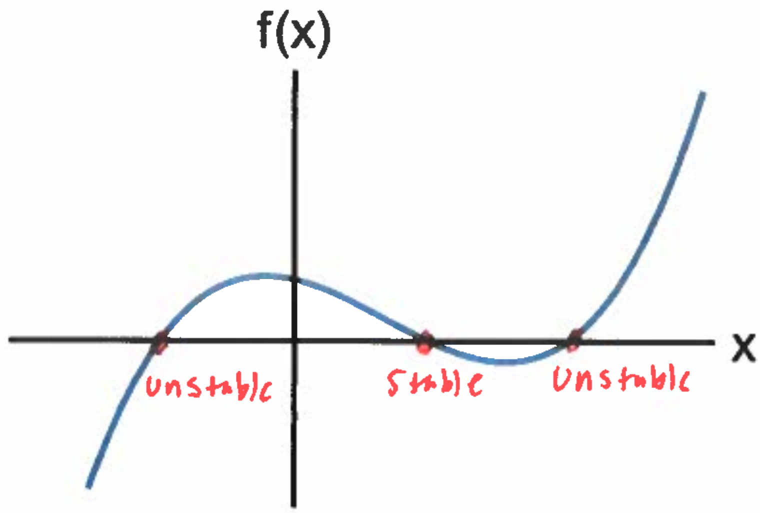
$$\lim_{t \to -\infty} x(t) = x^*.$$

   True.

2. (5 points) **(Short Answer:)** Consider the differential equation

$$\dot{x} = f(x)$$

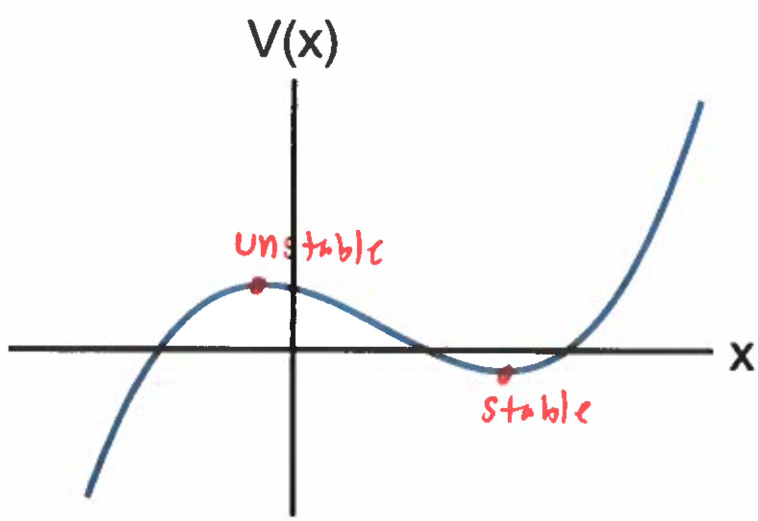
with $x \in \mathbb{R}$ for the function $f(x)$ drawn below. On the graph below, indicate the fixed points for this differential equation and their stability.



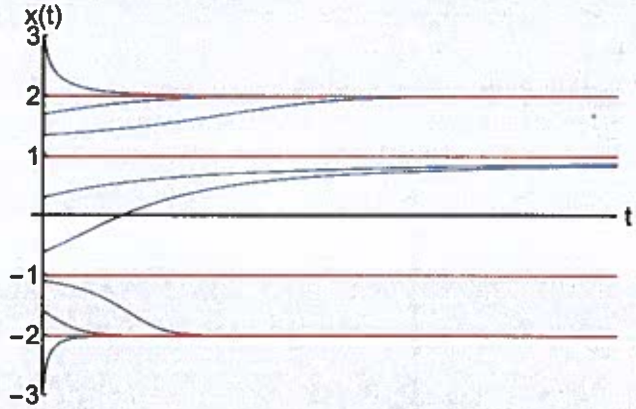
3. (5 points) **(Short Answer:)** Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

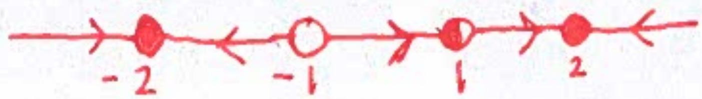
with $x \in \mathbb{R}$, where the potential $V(x)$ is drawn below. On the graph below, indicate the fixed points of this differential equation and their stability.



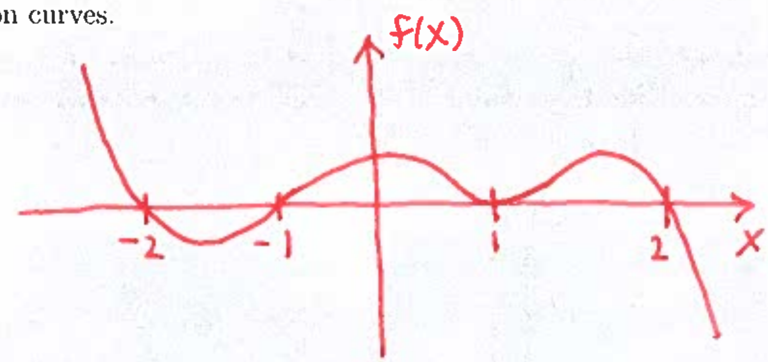
4. (15 points) The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$. The red curves indicate solution curves that do not change in time.



(a) (5 points) **Short Answer:** Sketch a one dimensional phase portrait that is consistent with the above solution curves.



(b) (5 points) **Short Answer:** Sketch a graph of $f(x)$ that is consistent with the above solution curves.



(c) (5 points) **Short Answer:** Give a formula for $f(x)$ that is consistent with these solution curves.

$$f(x) = -(x-2)(x-1)^2(x+1)(x+2)$$

5. (15 points) The growth of cancerous tumors can be modeled by the Gompertz law

$$\dot{N} = -a\ln\left(\frac{N}{b}\right)N.$$

where N is the number of cancerous cells and $a, b > 0$ are constant.

(a) (5 points) **Short Answer:** Determine the units of a and b.

$[a] = \frac{1}{\text{time}}$

$[b] = \text{cells}$

(b) (5 points) **Short Answer:** Give biological interpretations of a and b. **Hint:** Think about what happens if $N > b$, $N < b$ and $N = b$.

b = equilibrium density of cells

a = rate of growth of cells.

(c) (5 points) Show by choosing appropriate dimensionless variables x and τ that this system can be put into the following dimensionless form:

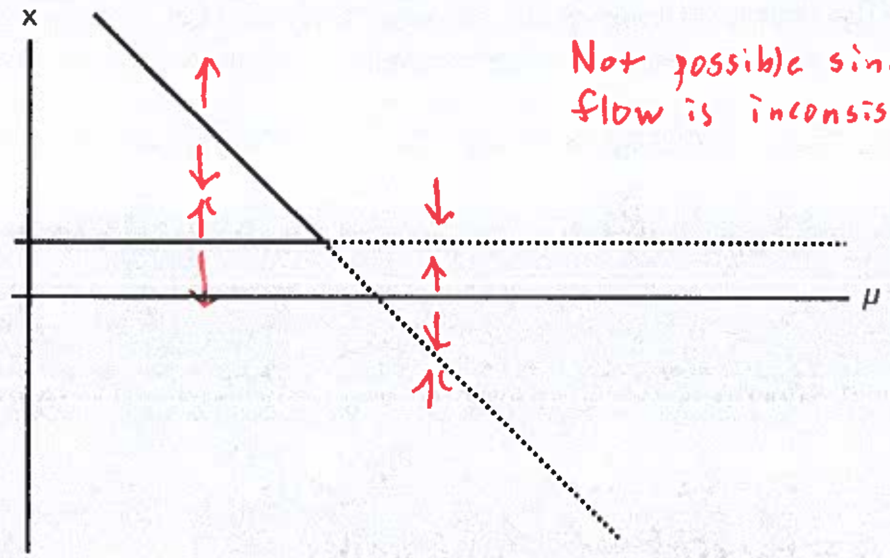
$$\frac{dx}{d\tau} = -\ln(x)x.$$

$x = \frac{N}{b}, \quad \tau = at$

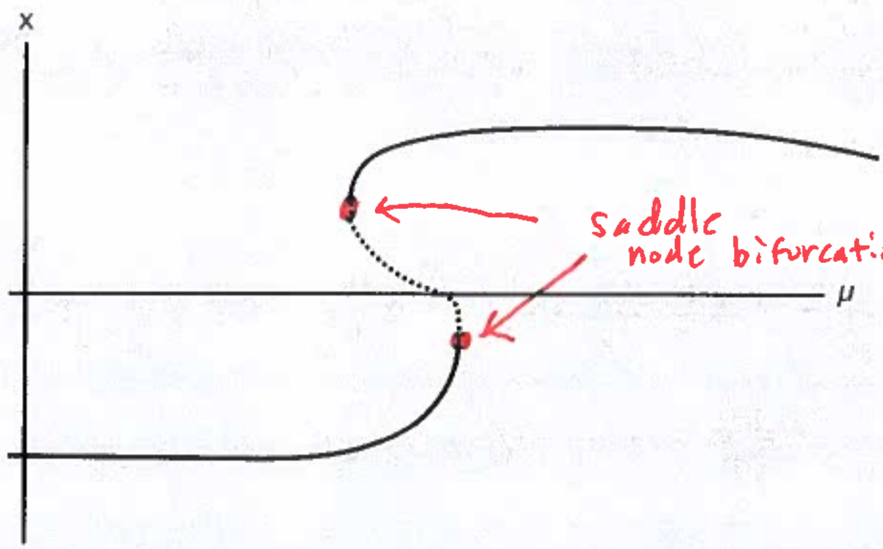
$\frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \frac{1}{ab}\dot{N} = \frac{1}{ab}\left(-a\ln\left(\frac{N}{b}\right)N\right) = \frac{1}{ab}\left(-a\ln(x)bx\right)$

$\Rightarrow \frac{dx}{d\tau} = -\ln(x)x.$

6. (10 points) Consider the following two potential bifurcation diagrams for a differential equation $\dot{x} = f(x;\mu)$, where stable or unstable fixed points are drawn as solid or dashed curves, respectively. For each diagram, circle all bifurcation points in the (μ, x) plane and classify what type of bifurcation they are or argue why the bifurcation diagram is impossible.



Not possible since the flow is inconsistent.



saddle node bifurcations.

7. (25 points) Consider the following dynamical system

$$\dot{x} = x(r - 2 - x),$$

where $r \in \mathbb{R}$ is a parameter.

(a) (5 points) Determine the fixed points for this problem.

$X = 0, \; x = r - 2$

(b) (5 points) Classify the stability of the fixed points for this problem as a function of r. You do not have to worry about the case when the fixed points are semistable.

$f'(0) = r - 2$

$\Rightarrow$ 0 is stable if $r < 2$ and unstable if $r > 2$

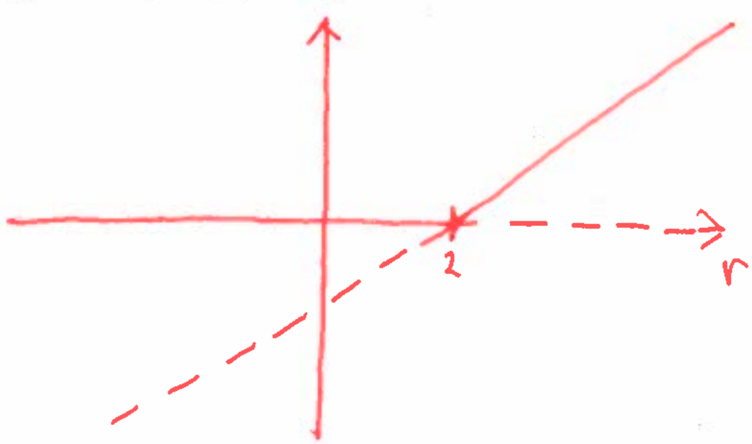
$\Rightarrow$ $r-2$ is unstable if $r > 2$ and unstable if $r < 2$

(c) (5 points) Sketch all qualitatively different phase portraits that occur as r is varied. You do not have to worry about the case when the fixed points are semistable.

Case 1: ($r < 2$)

Case 2: ($r > 2$)

(d) (10 points) Sketch a bifurcation diagram for this problem as a function of r and identify the types of bifurcations that occur.

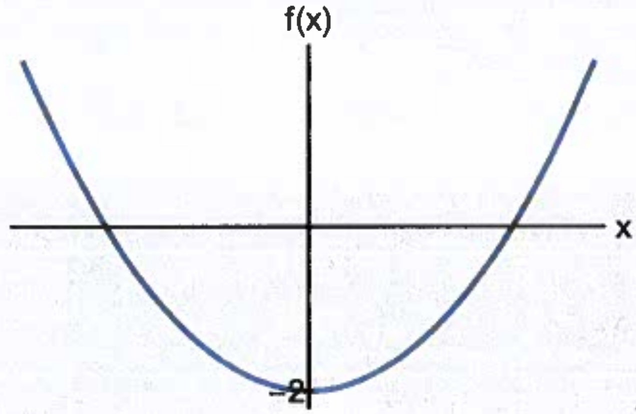


Transcritical bifurcation.

8. (15 points) Consider the following dynamical system

$$\dot{x} = rx + xf(x)$$

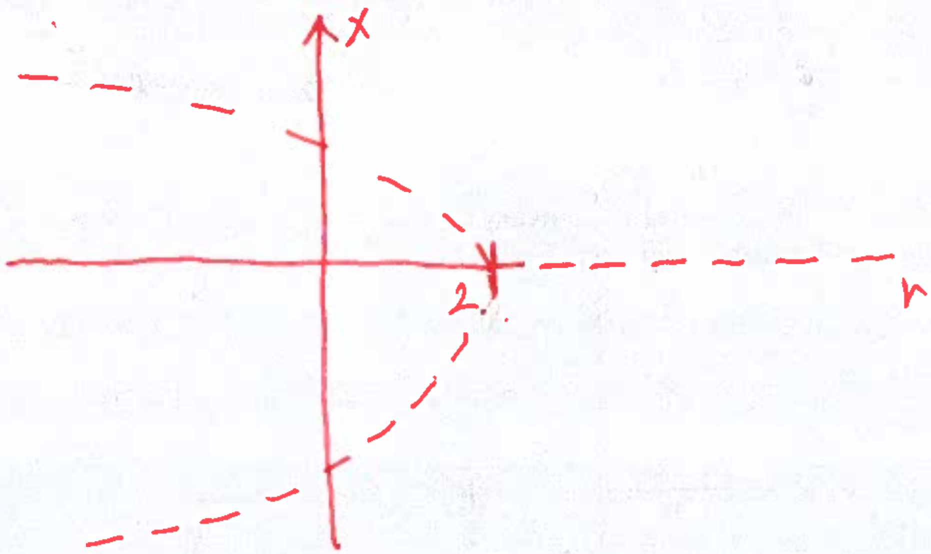
where $r \in \mathbb{R}$ is a parameter and $f : \mathbb{R} \mapsto \mathbb{R}$ is plotted below.



(a) (5 points) Sketch a phase portrait for this problem when $r = 0$.

$\lim_{x \to \infty} \dot{x} = \infty \Rightarrow$ rightmost fixed point is unstable

(b) (5 points) Sketch a bifurcation diagram for this problem as a function of r.



(c) (5 points) At what value of r, if any, do the bifurcations occur and what types of bifurcations occur.

At $r=2$ a pitchfork bifurcation occurs.