MTH 351/651
Fall 2022
Exam 2
10/28/22

Name (Print): Key

This exam contains 7 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit**. An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	5	
2	10	
3	10	
4	15	
5	15	
6	25	
7	20	
Total:	100	

Do not write in the table to the right.

1. (5 points) **Short Answer:** For the equation

$$\dot{\theta} = f(\theta),$$

what property must f satisfy to give a well-defined vector field on the circle?

f must be periodic with period 2π.

2. (10 points) Consider the following vector field on the circle

$$\dot{\theta} = \sin(\theta) - \cos(\theta).$$

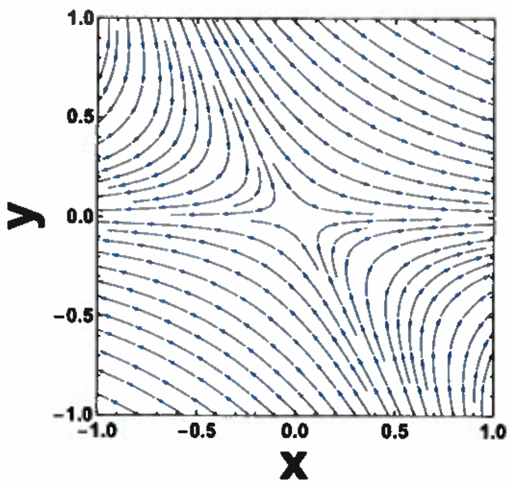
For this system find and classify all of the fixed points, and sketch the phase portrait on the circle.

There are fixed points when $\sin\theta = \cos\theta$ which implies $\theta = \pi/4, 5\pi/4$. Since $\dot{\theta}|_0 = -1$ it follows that $\theta = \pi/4$ is unstable and $5\pi/4 = \theta$ is stable.

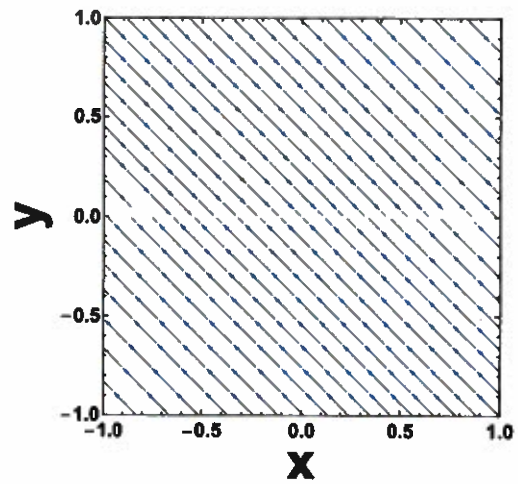
3. (10 points) Consider the following linear dynamical system

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix},$$

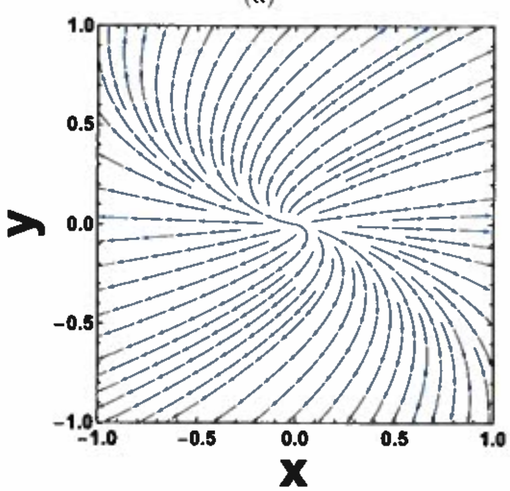
where A is a 2×2 matrix. Match the following possible phase portraits for this system with the corresponding matrices listed below. Note, there are more matrices than phase portraits, cross out any matrices that are not used. **Hint:** All of these matrices are upper triangular.



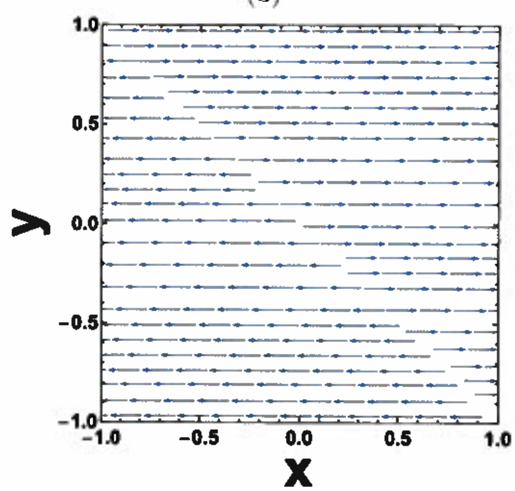
(a)



(b)



(c)



(d)

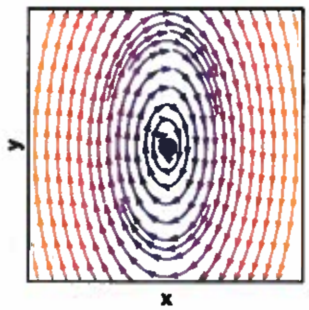
(c) $A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, $A = \cancel{\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}}$, (a) $A = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$, $A = \cancel{\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}}$,

$A = \cancel{\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}}$, (d) $A = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, $A = \cancel{\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}}$, (b) $A = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}$.

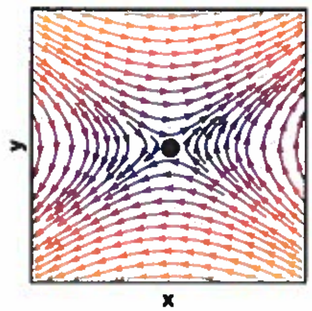
4. (15 points) For a system of differential equations of the form

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y),$$

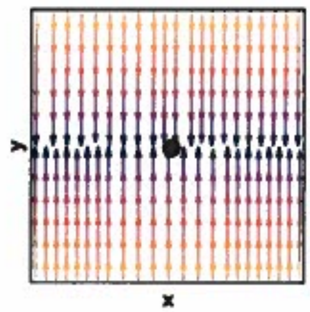
A fixed points is called **neutrally stable** if it is attracting but not Lyapunov stable. In the below phase portraits the origin is the only fixed point and the qualitative behavior of the phase portrait is the same for all of $\mathbb{R}^2$. **Note:** In the following questions, a phase portrait can satisfy multiple criterion for stability.



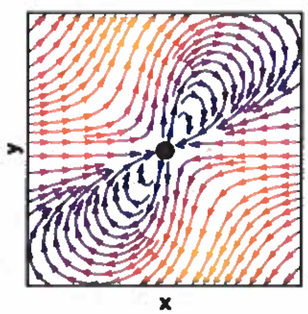
(a)



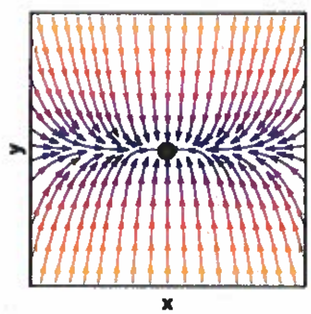
(b)



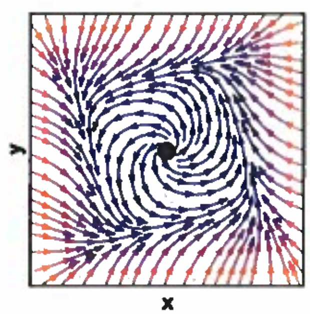
(c)



(d)



(e)



(f)

(a) (5 points) **Short Answer:** For which of these phase portraits is the origin **Lyapunov stable?**

(a), (c), (e)

(b) (5 points) **Short Answer:** For which of these phase portraits is the origin **asymptotically stable?**

(e)

(c) (5 points) **Short Answer:** For which of these phase portraits is the origin **neutrally stable?**

(d)

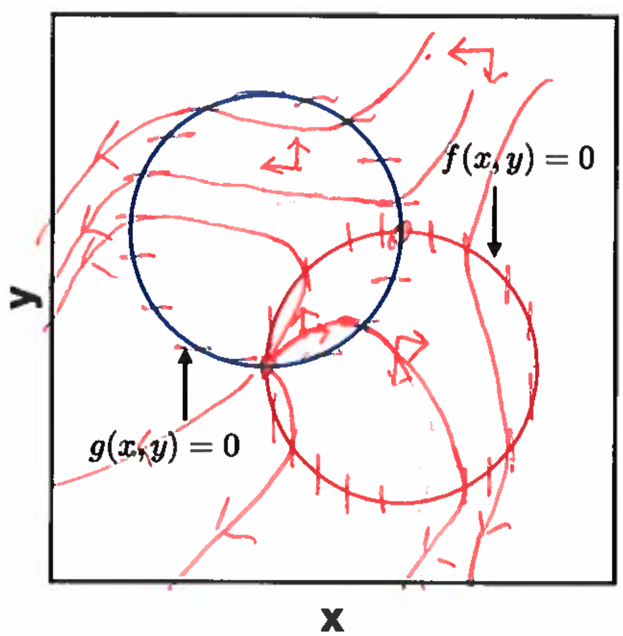
5. (15 points) Consider the system of differential equations

$$\frac{dx}{dt} = f(x,y),$$
$$\frac{dy}{dt} = g(x,y),$$

where f, g are continuous functions satisfying

$$\lim_{x\to\infty} f(x,y) = -\infty,$$
$$\lim_{y\to\infty} g(x,y) = -\infty.$$

The figure below is a plot of the curves satisfying $f(x,y) = 0$ and $g(x,y) = 0$. Using all of this information sketch a phase portrait for this system of differential equations on top of this diagram.



Your phase portrait must include the following to receive full credit

1. Clearly label all fixed points.
2. Draw enough solution trajectories so that you clearly illustrate all of the qualitatively different solution curves.

6. (25 points) The following system corresponds to a competing species model:

$$\dot{x} = x - xy,$$
$$\dot{y} = -y^2 + xy,$$

where $x, y \geq 0$ denote the populations of species X and species Y respectively.

(a) (5 points) **Short Answer:** Give a biological interpretation of this system. In particular, **briefly** explain how each species interacts with the other.

X- is a prey for y and is an herbavore
y- predator that is hostile to its own species.

(b) (15 points) Determine the nullclines, the fixed points for this system and analyze their stability using the Jacobian. What conclusions can you draw from your analysis of the Jacobian?

Nullclines.

$\dot{x} = 0 \Rightarrow x = 0$ or $y = 1$

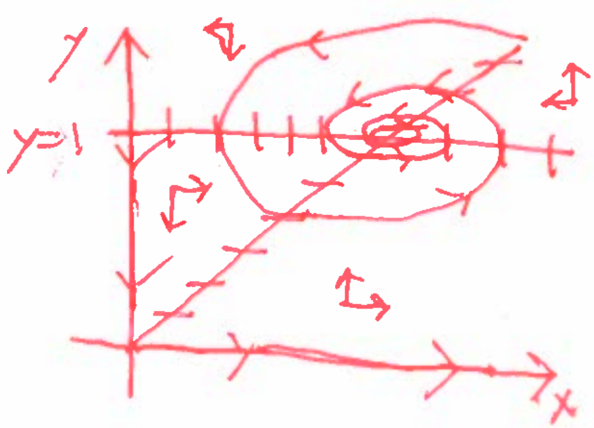
$\dot{y} = 0 \Rightarrow y = 0$ or $x = y$

The fixed points are $(0,0)$ and $(1,1)$. The Jacobian is given by

$$J(x,y) = \begin{bmatrix} 1-y & -x \\ y & -2y+x \end{bmatrix} \Rightarrow J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ (No information)}$$

$$J(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \text{ (Stable spiral)}$$

(c) (5 points) Sketch a phase portrait for this system **in the first quadrant** and in particular draw the flow on the lines $y = 0$ and $x = 0$. What does your phase portrait tell you about the long term behavior of these species?



The species coexist.

7. (20 points) Consider the following dynamical system

$$\dot{x} = -(x+y)(x^2+y^2)$$
$$\dot{y} = (x-y)(x^2+y^2).$$

(a) (10 points) Convert to polar coordinates to analyze the local stability of the origin.

(b) (10 points) Sketch the nullclines and a reasonable phase portrait for this system.

$$\dot{r} = \frac{x\dot{x}+y\dot{y}}{r} = \frac{-(x^2+y^2)(x^2+y^2)}{r} = -r^3$$

$$\dot{\theta} = \frac{x\dot{y}-\dot{y}x}{r^2} = \frac{(x^2+y^2)(x^2+y^2)}{r^2} = r$$