MTH 351/651
Homework #2

Due Date: September 09, 2022

1 Problems for Everyone

1. Consider the initial value problem:
$$\dot{x} = -x^{1/3},$$
$$x(0) = 0.$$

   (a) For $t_0 < 0$, show that the function
$$x(t) = \begin{cases} \left(\frac{2}{3}\right)^{3/2}(t_0 - t)^{3/2} & t \leq t_0 \\ 0 & t > t_0 \end{cases},$$
   is a differentiable function with a continuous derivative.

   (b) Show that for $t_0 < 0$, the function defined in part (a) is a solution to the initial value problem.

   (c) What does the existence of this family of solutions tell you about the uniqueness of solutions to this differential equation? Why does this result not contradict the existence and uniqueness theorem of differential equations?

2. A particle travels on the half line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

   (a) Find all values of c such that the origin $x = 0$ is a stable fixed point.

   (b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

3. D'Arcy Wentworth Thompson, a noted scientist of natural history, wrote in his book, *On Growth and Form* (1917): "But why, in the general run of shells, all the world over, in the past and in the present, one direction of twist is so overwhelmingly commoner than the other, nobody knows." Most snails species are dextral (right handed) in their shell pattern. Sinistral (left-handed) snails are exceedingly rare.

   (a) Let $p(t)$ be the ratio of dextral snails in the population of snails. Explain why
$$\dot{p} = rp(1-p)\left(p - \frac{1}{2}\right),$$
$$p(0) = p_0,$$
   is a plausible model for the dynamics of dextral snails if we assume $0 < p_0 < 1$ and $r > 0$.

(b) Sketch a phase portrait for this system.

(c) Suppose $p_0 \approx 1/2$. Explain in practical terms why this phase portrait justifies the observation that sinistral snails are rare. Explain why this is essentially a fluke and we could just as easily debating why dextral snails are rare.

4. In class we developed the logistic growth model of population growth:

$$\dot{P} = rP\left(1 - \frac{P}{\kappa}\right),$$

where $r > 0$ is a growth rate and $\kappa > 0$ is a carrying capacity. This model has unstable and stable fixed points at $P = 0$ and $P = \kappa$ respectively. Therefore, the model predicts that for for all positive initial conditions the population will reach equilibrium at the carrying capacity. However, this is somewhat unrealistic. Suppose we wanted to model population growth of humans and we start with $P_0 = 1$, i.e. there is only one human on the planet. Clearly the population would die out. In particular, for a sufficiently small initial population we would expect the population to die out. In this problem we are going to a develop a new mathematical model of the form

$$\dot{P} = F(P)$$

that corrects this problem in the logistic model.

(a) What properties should $F(P)$ satisfy in order to represent a realistic model of population growth? Your function must account for population extinction if P_0 is sufficiently small. Justify your answer.

(b) Sketch a graph of $F(P)$. Be sure to label everything that is important for the model.

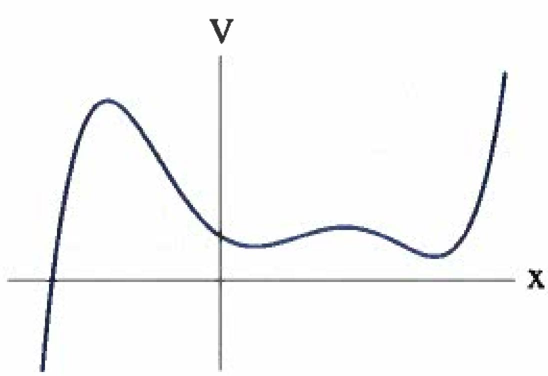
(c) Give a possible analytic formula for F that satisfies the properties you outlined above. With modeling you want to give the simplest possible example that works.

(d) Sketch a phase portrait for you system and discuss the consequences of this model.

5. Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

with $x \in \mathbb{R}$, where the potential $V(x)$ is drawn below. Sketch a phase portrait for this system.



Homework #2

#1.

Consider the intial value problem

$$\dot{x} = -x^{1/3},$$
$$x(0) = 0.$$

(a) For $t_0 < 0$, show that the function

$$x(t) = \begin{cases} (2/3)^{3/2}(t_0 - t)^{3/2}, & t \leq t_0, \\ 0, & t \geq t_0 \end{cases}$$

is a differentiable function with continuous derivative.

(b) Show that for $t_0 < 0$, the function defined in part (a) is a solution to the initial value problem.

(c) What does the existence of this family of solutions tell you about the uniqueness of solutions to this differential equation? Why does this result not contradict the existence and uniqueness theorem of differential equations.

Solution:

(a) $$\dot{x}(t) = \begin{cases} -(2/3)^{3/2}(3/2)(t_0 - t)^{1/2}, & t \leq t_0 \\ 0, & t \geq t_0 \end{cases}$$

which satisfies

$$\lim_{t \to t_0^-} \dot{x}(t) = \lim_{t \to t_0^+} \dot{x}(t)$$

and thus $x(t)$ is continuously differentiable.

(b) $$\dot{x}(t) = \begin{cases} -(2/3)^{1/2}(t_0 - t)^{1/2}, & t \leq t_0 \\ 0, & t > t_0 \end{cases}$$
$$= -x^{1/3}.$$

(c) Since $f(x) = -x^{1/3}$ is not differentiable at $x = 0$ it follows that solutions are not guaranteed to be unique. ∎

#2

A particle travels on the half line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

(a) Find all values of c such that $x=0$ is a stable fixed point.

(b) Now assume that c is chosen such that $x=0$ is stable. Can the particle ever reach the origin in finite time? Specifically, how long does it take for the particle to travel from $x=1$ to $x=0$, as a function of c?

Solution:

(a) Fixed points only exist if $c>0$. Since $f(x) = -x^c$ is negative for $c>0$ it follows that the fixed point is stable.

(b) If $c \geq 1$, $f(x)$ is differentiable and thus solutions cannot reach $x=0$ in finite time. If $0<c<1$ we have that the time of flight T satisfies

$$\int_1^0 -\frac{1}{x^c}\,dx = \int_0^T dt$$

$$\Rightarrow -\frac{x^{1-c}}{1-c}\Big|_1^0 = T$$

$$\Rightarrow T = \frac{1}{1-c}.$$

#3

(a) Let $p(t)$ be the ratio of dextral snails in the population of snails. Explain why

$$\dot{p} = rp(1-p)\left(p - \frac{1}{2}\right)$$

$$p(0) = p_0$$

is a plausible model for the dynamics of dextral snails if we assume $0<p_0<1$ and $r>0$.

(b) Sketch a phase portrait for this system.

(c) Suppose $p_0 \approx 1/2$. Explain in practical terms why this phase portrait justifies the observation that sinistral snails are rare. Explain why this is a fluke and we could be debating why dextral snails are rare.

Solution:

(a) We can express this system in the form:

$$\dot{p} = \hat{r}(p)\,p\,(1-p),$$

where $\hat{r}(p) = r(p - \tfrac{1}{2})$. That is, p undergoes logistic growth with a state dependent rate which is positive or negative depending on if there are more right handed then left handed snails.

(b)

(c) Even if $p_0 \approx \tfrac{1}{2}$ in the long run $\lim_{t\to\infty} p(t) = 0$ or $\lim_{t\to\infty} p(t) = 1$. Consequently, any imbalence in the population will lead to one type of species being selected. ∎

#4.

$$\dot{P} = F(P)$$

(a) What properties should $F(p)$ satisfy in order to represent a realistic model of population growth? Your function must account for population extinction if P_0 is sufficiently small.

(b) Sketch a graph of $F(p)$.

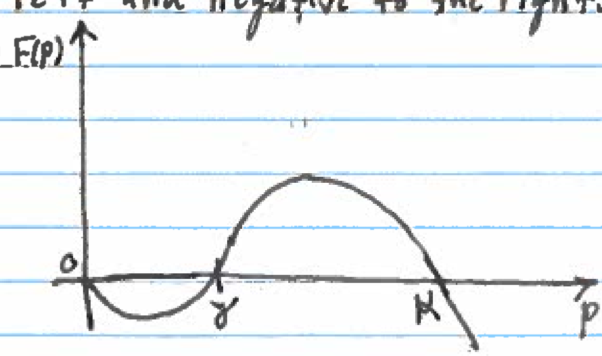
(c) Give a possible analytic formula for F that satisfies the properties outlined above.

(d) Sketch a phase portrait and discuss the consequences of the model.

Solution:

(a) Near 0, $F<0$ and near K F should be positive to the left and negative to the right.

(b)



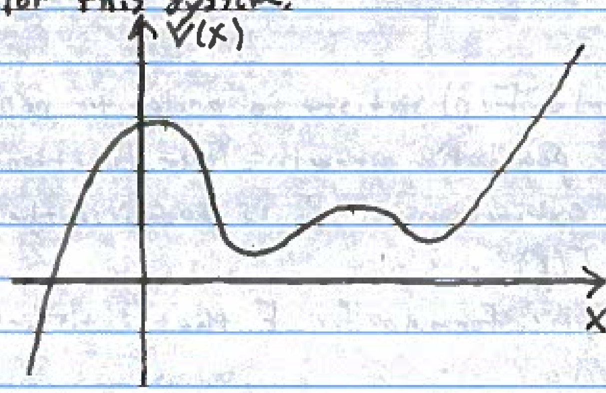
(c) One formula that works is
$$F(p) = rp(\gamma - p)(p - K),$$
where $r > 0$. Note, the r is necessary to balance units.

(d) 

The population can go extinct if the initial population is too small.

#5

Consider the differential equation
$$\dot{x} = -\frac{dV}{dx},$$
with $x \in \mathbb{R}$, where $V(x)$ is drawn below. Sketch a phase portrait for this system.



Solution: