MTH 351/651
Homework #1

Due Date: September 02, 2022

1 Problems for Everyone

1. Consider the system $\dot{x} = \sin(x)$.
   (a) Find all fixed points of the flow.
   (b) At which points x does the flow have the greatest velocity to the right?
   (c) Find the flows acceleration $\ddot{x}$ as a function of x.
   (d) Find the points where the flow has maximum positive acceleration.

2. For the following equations sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions. You must include enough sketches of $x(t)$ to illustrate all qualitatively different solution solution curves.
   (a) $\dot{x} = 1 - x^{14}$.
   (b) $\dot{x} = e^{-x}\sin(x)$.
   (c) $\dot{x} = 1 - 2\cos(x)$.
   (d) $\dot{x} = e^{x} - \cos(x)$ (You won't be able to find the fixed points explicitly, but you can still determine the qualitative behavior).

3. pg. 37: #2.2.8, 2.2.9, 2.2.10.

4. The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to air resistance.
   (a) Obtain the analytic solution for $v(t)$, assuming that $v(0) = 0$.
   (b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.
   (c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

5. Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a, b > 0$. Let $x(t) = X(t)/(X(t) + Y(t))$ denotes X's share of the total population.
   (a) Show that $\dot{x} = (a - b)x(1 - x)$.
   (b) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \to \infty$. What does this result imply about the population?
   (c) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \to \infty$. What does this result imply about the population?
   (d) What happens if $a = b$?