

MTH 351/651

Homework #2

Due Date: September 09, 2022

1 Problems for Everyone

1. Consider the initial value problem:

$$\begin{aligned}\dot{x} &= -x^{1/3}, \\ x(0) &= 0.\end{aligned}$$

- (a) For $t_0 < 0$, show that the function

$$x(t) = \begin{cases} \left(\frac{2}{3}\right)^{3/2} (t_0 - t)^{3/2} & t \leq t_0 \\ 0 & t > t_0 \end{cases},$$

is a differentiable function with a continuous derivative.

- (b) Show that for $t_0 < 0$, the function defined in part (a) is a solution to the initial value problem.
- (c) What does the existence of this family of solutions tell you about the uniqueness of solutions to this differential equation? Why does this result not contradict the existence and uniqueness theorem of differential equations?
2. A particle travels on the half line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.
- (a) Find all values of c such that the origin $x = 0$ is a stable fixed point.
- (b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c ?

3. D'Arcy Wentworth Thompson, a noted scientist of natural history, wrote in his book, *On Growth and Form* (1917): "But why, in the general run of shells, all the world over, in the past and in the present, one direction of twist is so overwhelmingly commoner than the other, nobody knows." Most snails species are dextral (right handed) in their shell pattern. Sinistral (left-handed) snails are exceedingly rare.

- (a) Let $p(t)$ be the ratio of dextral snails in the population of snails. Explain why

$$\begin{aligned}\dot{p} &= rp(1-p) \left(p - \frac{1}{2}\right), \\ p(0) &= p_0,\end{aligned}$$

is a plausible model for the dynamics of dextral snails if we assume $0 < p_0 < 1$ and $r > 0$.

- (b) Sketch a phase portrait for this system.
- (c) Suppose $p_0 \approx 1/2$. Explain in practical terms why this phase portrait justifies the observation that sinistral snails are rare. Explain why this is essentially a fluke and we could just as easily debating why dextral snails are rare.
4. In class we developed the logistic growth model of population growth:

$$\dot{P} = rP \left(1 - \frac{P}{\kappa} \right),$$

where $r > 0$ is a growth rate and $\kappa > 0$ is a carrying capacity. This model has unstable and stable fixed points at $P = 0$ and $P = \kappa$ respectively. Therefore, the model predicts that for all positive initial conditions the population will reach equilibrium at the carrying capacity. However, this is somewhat unrealistic. Suppose we wanted to model population growth of humans and we start with $P_0 = 1$, i.e. there is only one human on the planet. Clearly the population would die out. In particular, for a sufficiently small initial population we would expect the population to die out. In this problem we are going to develop a new mathematical model of the form

$$\dot{P} = F(P)$$

that corrects this problem in the logistic model.

- (a) What properties should $F(P)$ satisfy in order to represent a realistic model of population growth? Your function must account for population extinction if P_0 is sufficiently small. Justify your answer.
- (b) Sketch a graph of $F(P)$. Be sure to label everything that is important for the model.
- (c) Give a possible analytic formula for F that satisfies the properties you outlined above. With modeling you want to give the simplest possible example that works.
- (d) Sketch a phase portrait for you system and discuss the consequences of this model.
5. Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

with $x \in \mathbb{R}$, where the potential $V(x)$ is drawn below. Sketch a phase portrait for this system.

