MTH 351/651
Homework #3

Due Date: September 16, 2022

1 Problems for Everyone

1. Consider the following dynamical system

$$\dot{x} = ax - x^3,$$

where a is a real number that can be positive, negative, or zero.

   (a) For all three cases find the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

   (b) For all three cases, calculate and plot the potential function $V(x)$.

2. The simplest model of malaria assumes that the mosquito population is at equilibrium and models the number of humans infected by malaria, $I(t)$, by the following differential equation:

$$\dot{I} = \frac{\alpha\mu I}{\alpha I + r}(N - I) - \beta I,$$

where $N > 0$ is the total population of healthy and infected individuals and is assumed constant and α, β, r, μ are positive parameters.

   (a) Determine the units of α, β, r and μ.

   (b) Show that there exists a dimensionless change of variables in I and τ such that this system is equivalent to the following dimensionless system

$$\frac{dx}{d\tau} = \frac{Ax}{x + B}(1 - x) - x,$$

   where $A, B > 0$ are dimensionless constants.

   (c) In this dimensionless system the rate of infection is given by

$$f(x) = \frac{Ax}{x + B}.$$

   Calculate $\lim_{x\to\infty} f(x)$, $f'(0)$, and sketch a generic graph of $f(I)$. What effect does changing the parameters A and B have on the graph?

   (d) Calculate the fixed points for this system and analyze there stability.

   (e) Determine the threshold criteria for this model to have an endemic equilibrium.

3. For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. Sketch a bifurcation diagram of fixed points x^* versus r. In each bifurcation diagram determine what type of bifurcation occurs.
   (a) $\dot{x} = 1 + rx + x^2$.
   (b) $\dot{x} = rx + x^2$.
   (c) $\dot{x} = r - \cosh(x)$.
   (d) $\dot{x} = x - rx(1 - x)$.
   (e) $\dot{x} = x + \dfrac{rx}{1 + x^2}$.
   (f) $\dot{x} = r - 3x^2$.
   (g) $\dot{x} = rx - \dfrac{x}{1 + x^2}$.
   (h) $\dot{x} = rx + \dfrac{x^3}{1 + x^2}$.

4. Consider the system $\dot{x} = rx + x^3 - x^5$, where $r \in \mathbb{R}$ is a parameter.
   (a) Find algebraic expression for all of the fixed points as r is varied.
   (b) Sketch all possible phase portraits as r is varied.
   (c) Sketch the bifurcation diagram for this problem. In this bifurcation diagram, determine what types of bifurcations occur.
   (d) Calculate the explicit values for all of the bifurcation points in this problem.

5. Consider a gene that is activated by the presence of a biochemical substance S. Let $g(t)$ denote the concentration of the gene product at time t, and assume that the concentration of S, denoted by s_0, is fixed. A model describing the dynamics of g is as follows:

$$\frac{dg}{dt} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2},$$

   where the k's are positive constants.
   (a) Determine the units of each of the k's.
   (b) Show that this equation can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2},$$

   where $r > 0$ and $s \geq 0$ are dimensionless constants.
   (c) In the case when $s = 0$, sketch a bifurcation diagram in the parameter r.
   (d) In the case when $r = .4$, qualitatively sketch a bifurcation diagram in s. Hint, plot $\frac{dx}{d\tau}$ vs x for $s = 0$ and $r = .4$ and think about how this plot qualitatively changes as s is varied.
   (e) Assume that initially there is no gene product, that is $x(0) = 0$, $s = 0$, $r = .4$, and s is slowly increased from zero, i.e. the biochemical substance s is slowly introduced. What happens to $x(\tau)$? Why? What happens if s goes back to zero? Does the gene turn off again?