MTH 351/651
Homework #6

Due Date: October 12, 2022

1 Problems for Everyone

1. Consider the linear system

$$\dot{x} = x - y,$$
$$\dot{y} = x + y.$$

   (a) Show that the system has eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

   (b) Express the generic solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

   as a sum of two real-valued functions. **Hint:** First, express $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Second, use the fact that for $\omega \in \mathbb{R}$, $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ to rewrite the solution $\mathbf{x}(t)$ in terms of sines and cosines and then separate the terms that have a prefactor of i from those that do not. Third, find conditions on a_1, a_2, b_1, b_2 that ensure the terms involving i vanish.

2. For the following systems, find the nullclines and the fixed points and plot a plausible phase portrait for the system. You do not have to analyze the stability of the fixed points.

   (a) $\dot{x} = x - x^3$ and $\dot{y} = -y$.

   (b) $\dot{x} = y$ and $\dot{y} = x(1 + y) - 1$.

   (c) $\dot{x} = \sin(y)$ and $\dot{y} = x - x^3$.

   (d) $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.