MTH 351/651
Homework #7

Due Date: October 21, 2022

1 Problems for Everyone

1. Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.
   (a) Find all the fixed points and classify them.
   (b) Show that the line $x = y$ is invariant, i.e. any trajectory that starts on it stays on it.
   (c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all trajectories.
   (d) Sketch the phase portrait.
   (e) Use the StreamPlot command in Mathematica to plot an accurate phase portrait on the square domain $-20 \le x, y \le 20$. Notice the trajectories seem to approach a curve as $t \to -\infty$; explain this curve intuitively and find an approximate equation for the curve. You do not have to submit your Mathematica code or any output of your code.

2. In this problem we study the competing species model:

$$\begin{cases} \dot{x} = ax + bxy \\ \dot{y} = cy + dxy \end{cases},$$

   where $a, b, c, d \in \mathbb{R}$. In the following cases, sketch the phase portrait, analyze the stability of any fixed points, and give a biological interpretation for each case.
   (a) $a, b, c, d > 0$.
   (b) $a, b, c > 0$ and $d < 0$.
   (c) $a, b, d > 0$ and $c < 0$.
   (d) $a, b > 0$ and $c, d < 0$.
   (e) $b, c > 0$ and $a, d < 0$.
   (f) $a, c > 0$ and $b, d < 0$.
   (g) $b, d > 0$ and $a, c < 0$.
   (h) $b > 0$ and $a, c, d < 0$.
   (i) $a > 0$ and $b, c, d < 0$.
   (j) $a, b, c, d < 0$.

3. Consider the following model for the interaction of the population of deer N_1 and rabbits N_2:

$$\begin{cases} \dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{\kappa_1}\right) - \alpha N_1 N_2 \\ \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_1}{\kappa_2}\right) - \beta N_1 N_2 \end{cases},$$

   where $r_1, r_2, \kappa_1, \kappa_2, \alpha, \beta$ are constants.

(a) Give biological interpretations of each of the parameters.

(b) Nondimensionalize this system. There are many ways to do this. You should do the most natural one that makes sense biologically and makes the problem as simple as possible.

(c) Classify the fixed points for this system and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

4. Consider the system $\dot{x} = xy$, $\dot{y} = x^2 - y$.

   (a) Show that the linearization predicts the origin is a non-isolated fixed point.

   (b) Show in fact that the origin is an isolated fixed point.

   (c) Sketch the phase portrait for this system.

   (d) Is the origin repelling, attracting, a saddle, or what?

5. Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x$. Prove that the origin is a spiral, although the linearization predicts a center.

6. The system $\dot{x} = xy - x^2y + y^3$, $\dot{y} = y^2 + x^3 - xy^2$ has a fixed point at the origin that is difficult to analyze. Sketch the phase portrait for this system. Converting to polar coordinates could be useful.