MTH 351/651
Homework #8

Due Date: November 11, 2022

1 Problems for Everyone

1. For the following conservative systems find all the equilibrium points and classify them, find a conserved quantity, sketch the phase portrait, find explicit formulas for any homoclinic or heteroclinic orbits.

   (a) $\ddot{x} = x^3 - x$,

   (b) $\ddot{x} = x - x^2$.

2. The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2,$$

   where $u = 1/r$ and r, θ are polar coordinates of the planet in its plane of motion. The parameter α is positive and εu^2 is Einstein's correction. Here ε is a very small positive number.

   (a) Rewrite the system in the (u, v) phase plane, where $v = \frac{du}{d\theta}$.

   (b) Find all the equilibrium points of the system.

   (c) Show that one of the equilibrium is a center in the (u, v) phase plane, according to the linearization. Is it a nonlinear center?

   (d) Show that the equilibrium point found in (c) corresponds to a circular planetary orbit.

3. The Duffing oscillator is described by the following differential equation

$$\ddot{x} + x + \varepsilon x^3 = 0.$$

   (a) Show that this system has a nonlinear center at the origin for $\varepsilon > 0$.

   (b) If $\varepsilon < 0$, show that all trajectories near the origin are closed. What about trajectories that are far from the origin?

4. For each of the following systems, locate the fixed points and calculate the index.

   (a) $\dot{x} = x^2$, $\dot{y} = y$

   (b) $\dot{x} = y - x$, $\dot{y} = x^2$

   (c) $\dot{x} = y^3$, $\dot{y} = x$

   (d) $\dot{x} = xy$, $\dot{y} = x + y$

5. A closed orbit in the phase plane encircles S saddles, N nodes, F spirals, and C centers, all of the usual type. Show that

$$N + F + C = 1 + S.$$

6. A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say C_1 and C_2, lie inside the third cycle C_3. However, C_1 does not lie inside C_2, nor vice-versa.

   (a) Sketch the arrangement of the three cycles.

   (b) Show that there must be at least one fixed point in the region bounded by C_1, C_2, and C_3.

7. Consider a smooth vector field $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ on the plane, and let C be a simple closed curve that does not pass through any fixed points. As usual, let $\phi = \tan^{-1}(\dot{y}/\dot{x})$.

   (a) Show that $d\phi = (f dg - g df)/(f^2 + g^2)$.

   (b) Derive the following integral formula for the index

$$I_C = \frac{1}{2\pi} \int_C \frac{f dg - g df}{f^2 + g^2}.$$

8. Consider the following system of differential equations

$$\dot{x} = x - y - x(x^2 + 5y^2),$$
$$\dot{y} = x + y - y(x^2 + y^2).$$

   (a) Classify the fixed point at the origin.

   (b) Rewrite the system in polar coordinates.

   (c) Determine the maximum radius centered at the origin such that all trajectories have a radially outward component on it.

   (d) Determine the circle of minimum radius centered at the origin such that all trajectories have a radially inward component on it.

   (e) Prove that the system has a limit cycle.

9. Consider the system

$$\dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x),$$
$$\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x).$$

   (a) Show that the origin is an unstable fixed point.

   (b) By considering $\dot{V}$, where $V = (1 - 4x^2 - y^2)^2$, show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.