Lecture 13: Phase Portraits and Linearization

Example:

Sketch the phase portrait of

$$\begin{cases} \dot{x} = x + e^{-y} = f(x,y) \\ \dot{y} = -y \quad = g(x,y) \end{cases}$$

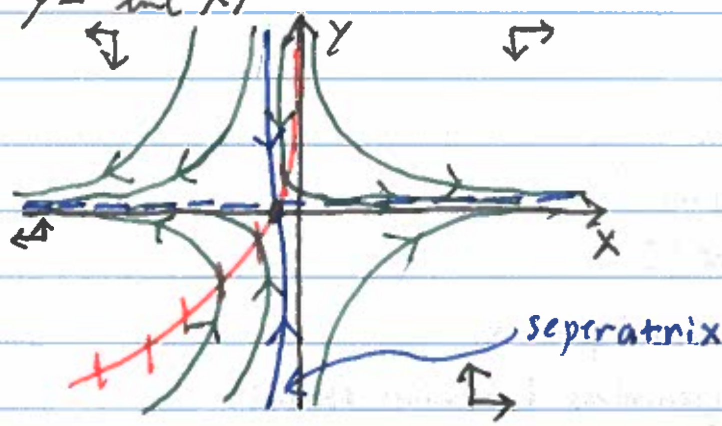
Nullclines

N1 ($\dot{x}=0$):

$$x + e^{-y} = 0$$
$$\Rightarrow e^{-y} = -x$$
$$\Rightarrow y = -\ln(-x)$$

N2 ($\dot{y}=0$):

$$y = 0$$



Linearize Around $(-1,0) = \vec{x}^* = (x^*, y^*)$

$$\dot{x} = f(\vec{x}^*)^{\,0} + \frac{\partial f}{\partial x}\Big|_{\vec{x}^*}(x - x^*) + \frac{\partial f}{\partial y}\Big|_{\vec{x}^*}(y - y^*) + \text{h.o.t.}$$
$$\dot{y} = g(\vec{x}^*)^{\,0} + \frac{\partial g}{\partial x}\Big|_{\vec{x}^*}(x - x^*) + \frac{\partial g}{\partial y}\Big|_{\vec{x}^*}(y - y^*) + \text{h.o.t.}$$

Change variables:

$$\bar{x} = (x - x^*), \quad \bar{y} = (y - y^*)$$
$$\Rightarrow \dot{\bar{x}} = \nabla f|_{\vec{x}^*} \cdot \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$
$$\dot{\bar{y}} = \nabla g|_{\vec{x}^*} \cdot \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$
$$\Rightarrow \dot{\bar{\vec{x}}} = J(\vec{F})\Big|_{\vec{x}^*} \bar{\vec{x}}, \quad J(F) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \text{Jacobian Matrix.}$$

In this case

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -e^{-y}$$

$$\frac{\partial g}{\partial x} = 0, \quad \frac{\partial g}{\partial y} = -1$$

$$\Rightarrow J(\vec{F})\big|_{(-1,0)} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 1, \ \lambda_2 = -1 \Rightarrow \text{saddle}$$

Phase Plane Summary:

1. Draw nullclines
2. Draw direction arrows
3. Find equilibrium
4. Calculate Jacobian
5. Determine eigenvalues

$Re(\lambda_1), Re(\lambda_2) \neq 0$:

- hyperbolic fixed points
- sign of eigenvalues determines stability
  - $\Rightarrow Re(\lambda_1), Re(\lambda_2) < 0 \Rightarrow$ asymptotic stability

$Re(\lambda_1) = 0$ or $Re(\lambda_2) = 0$:

- Stability cannot be determined
- more work is needed

Example:

$$\dot{x} = x(3-x) - 2xy = f(x,y)$$
$$\dot{y} = y(2-y) - xy = g(x,y)$$

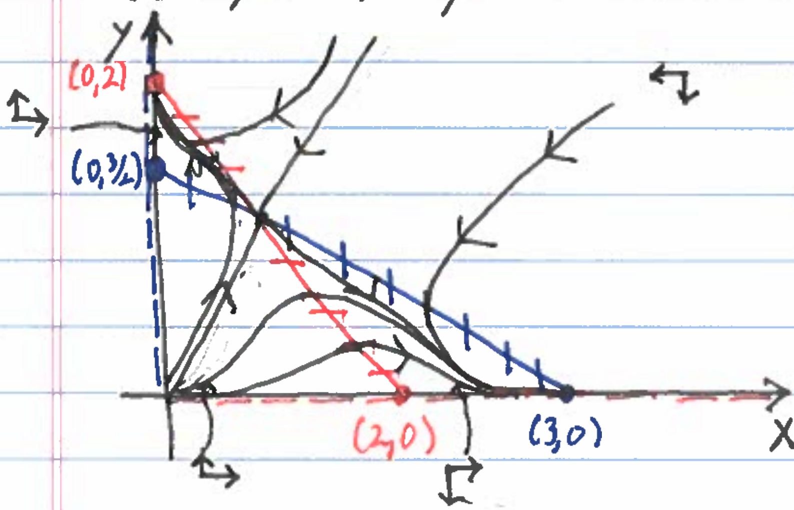
Nullclines

N1: $x = 0$ $(\dot{x}=0)$

N2: $y = -\frac{1}{2}x + \frac{3}{2}$ $(\dot{x}=0)$

N3: $y = 0$ $(\dot{y}=0)$

N4: $y = 2 - x$ $(\dot{y}=0)$



Fixed point at (0,2), (3,0) and when $y = -\frac{1}{2}x + \frac{3}{2}$ intersects with $y = 2 - x$.

$$\Rightarrow 2 - x = -\frac{1}{2}x + \frac{3}{2}$$
$$\Rightarrow 4 - 2x = -x + 3$$
$$x = 1, \quad y = 1.$$

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{bmatrix}$$

$$\Rightarrow J(0,2)=\begin{bmatrix}-1 & 0\\ -2 & -2\end{bmatrix} \Rightarrow \lambda_1=-1,\ \lambda_2=-2 \rightarrow$$ Asymptotically stable fixed point.

$$J(3,0)=\begin{bmatrix}-3 & -6\\ 0 & -1\end{bmatrix} \Rightarrow \lambda_1=-3,\ \lambda_2=-1 \Rightarrow$$ Asymptotically stable fixed point.

$$J(1,1)=\begin{bmatrix}-1 & -2\\ -1 & -1\end{bmatrix} \Rightarrow \lambda_{1,2}=\frac{2\pm\sqrt{1-4(-1)}}{2}=\frac{2\pm\sqrt{5}}{2} \Rightarrow$$ saddle.