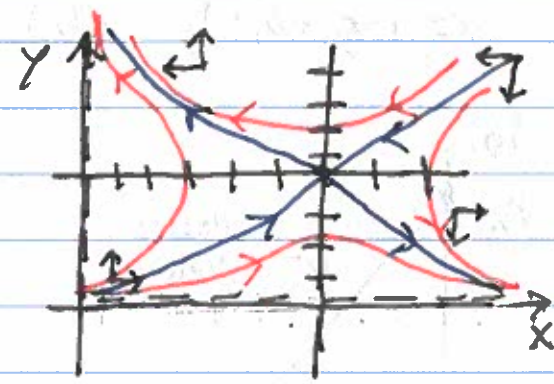
Lecture 14: Lotka Volterra Systems

Competition -

$\dot{x} = r_x x - a_x xy$, species x and y have infinite resources
$\dot{y} = r_y y - a_y xy$ but are hostile to each other.

Nullclines



1. $\dot{x} = 0 \Rightarrow x = 0,\ y = r_x/a_x$
2. $\dot{y} = 0 \Rightarrow y = 0,\ x = r_y/a_y$

Jacobian

$$J = \begin{bmatrix} r_x - a_x y & -a_x x \\ -a_y y & r_y - a_y x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} r_x & 0 \\ 0 & r_y \end{bmatrix} \Rightarrow \lambda_1 = r_x,\ \lambda_2 = r_y \text{ origin is unstable.}$$

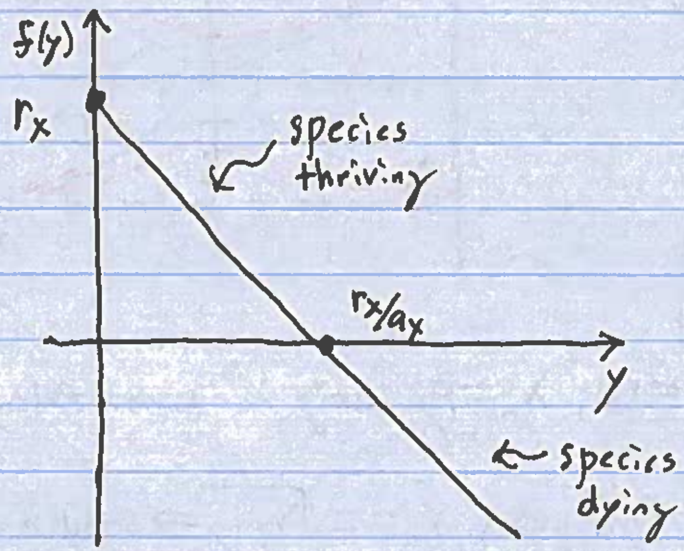
$$J(r_y/a_y, r_x/a_x) = \begin{bmatrix} 0 & -r_y a_x/a_y \\ -r_x a_y/a_x & 0 \end{bmatrix} \Rightarrow \lambda_1, \lambda_2 = \pm\sqrt{r_x r_y} \Rightarrow \text{saddle.}$$

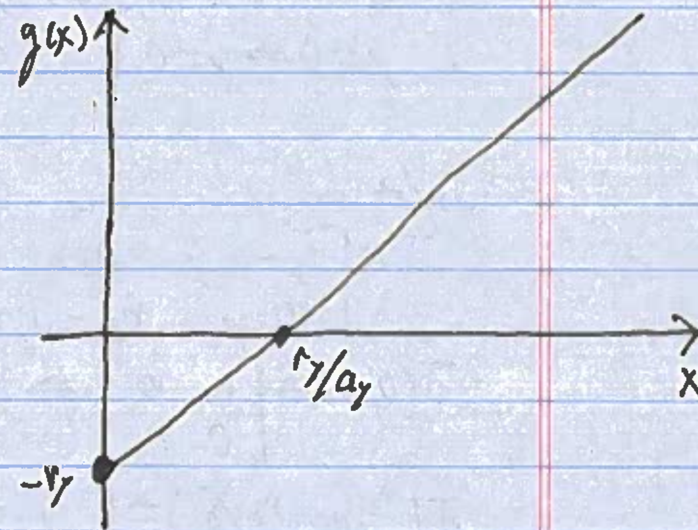
Hartmann-Grobmann Theorem - If the Jacobian at a fixed point $\vec{x}^*$ has eigenvalues with nonzero imaginary part, then $\dot{\vec{x}} = F(\vec{x})$ is locally equivalent to $\dot{\vec{x}} = J(\vec{x})$ near $\vec{x}^*$.

Predator-Prey -

$\dot{x} = r_x x - a_x x y$, species x grows without bound without species y
$\dot{y} = -r_y y + a_y x y$ but is preyed upon by species y. Species y undergoes pure death without species x and preys on species x.

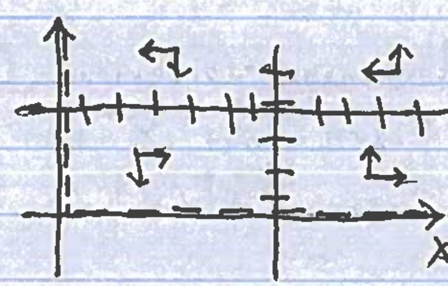
$\Rightarrow \dot{x} = (r_x - a_x y)x = f(y)x$ $\Rightarrow$ f, g are species dependent growth rates.
$\dot{y} = (-r_y + a_y x)y = g(x)y$





Nullclines:

1. $\dot{x} = 0$, $x = 0$, $y = r_x/a_x$
2. $\dot{y} = 0$, $y = 0$, $x = r_y/a_y$



$$J(x,y) = \begin{bmatrix} r_x - a_x y & -a_x x \\ a_y y & -r_y + a_y x \end{bmatrix}$$

$$\Rightarrow J(r_y/a_y, r_x/a_x) = \begin{bmatrix} 0 & -r_y a_y/a_x \\ r_x a_y/a_x & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = i\sqrt{r_x r_y} \Rightarrow$$ The linearization tells us nothing.

* We need to do more work to deduce stability.

For this problem we have:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(-r_y + a_y x)y}{(r_x - a_x y)x}$$

$$\Rightarrow \frac{(r_x - a_x y)}{y}\frac{dy}{dx} = \frac{(-r_y + a_y x)}{x}$$

$$\Rightarrow \int\left(\frac{r_x}{y} - a_x\right)dy = \int\left(\frac{-r_x}{x} + a_y\right)dx$$

$$\Rightarrow r_x \ln(y) - a_x y = -r_y \ln(x) + a_y x + E$$

$$\Rightarrow \quad E(x,y) = r_x \ln(y) + r_y \ln(x) - a_x y - a_y x$$

level sets of this function correspond to solution curves