Lecture 15: Polar Coordinates

Example: r^2 ← distance from origin squared.

$$\dot{x} = -y + (x^2+y^2)(1-x^2-y^2)x$$
$$\dot{y} = x + (x^2+y^2)(1-x^2-y^2)y$$

linear terms   nonlinear terms

$(0,0)$ is a fixed point.

$$\Rightarrow J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow \lambda_{1,2} = \pm i \Rightarrow$ More work is needed.

Convert to polar coordinates:

$$x = r\cos\theta, \quad r^2 = x^2+y^2$$
$$y = r\sin\theta \quad \theta = \tan^{-1}(y/x)$$

$$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}, \quad \dot{\theta} = \frac{1}{1+y^2/x^2}\frac{x\dot{y}-y\dot{x}}{x^2}$$
$$\Rightarrow \dot{r} = \frac{x\dot{x}+y\dot{y}}{r}, \quad \dot{\theta} = \frac{x\dot{y}-y\dot{x}}{r^2}$$

Therefore,

$$\dot{r} = \frac{-xy + (x^2+y^2)(1-x^2-y^2)x^2 + yx + (x^2+y^2)(1-x^2-y^2)y^2}{r}$$
$$= \frac{r^4(1-r^2)}{r} = r^3(1-r^2)$$

$r=1$ is an invariant set.

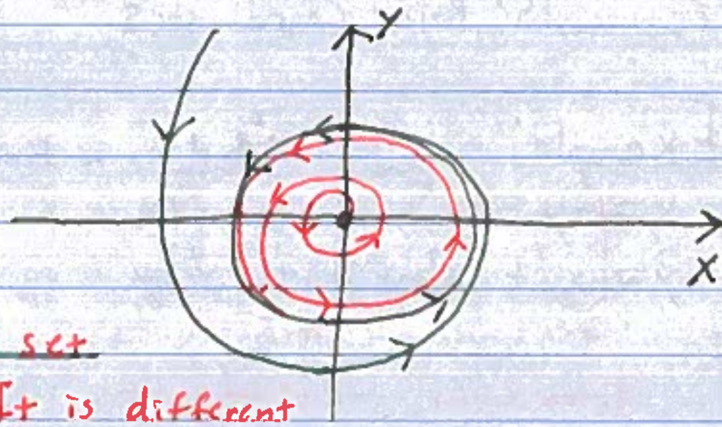
$$\dot{\theta} = \frac{x^2 + (x^2+y^2)(1-x^2-y^2)xy + y^2 - (x^2+y^2)(1-x^2-y^2)xy}{r^2}$$
$$= 1.$$

Note:

$r<1 \Rightarrow \frac{dr}{dt}>0$

$r=1 \Rightarrow \frac{dr}{dt}=0$

$r>1 \Rightarrow \frac{dr}{dt}<0$

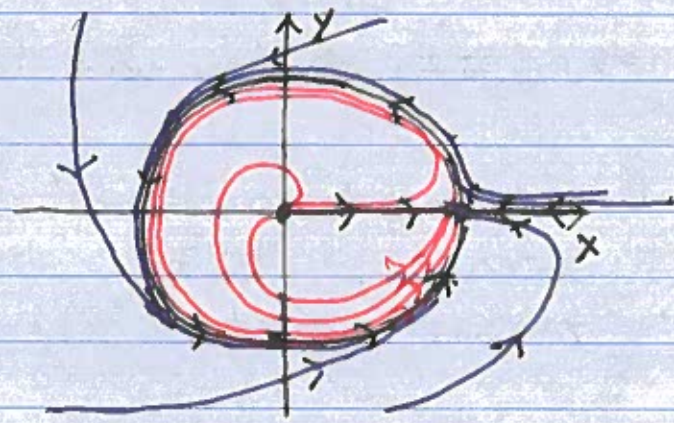


- $r=1$ is a new invariant set called a limit cycle. It is different from a nonlinear center since it is isolated.

Example:

$\dot{r}=r(1-r)$

$\dot{\theta}=\sin^2(\theta/2)$



$x=r\cos\theta$

$y=r\sin\theta$

$\Rightarrow \dot{x}=\dot{r}\cos\theta-r\sin\theta\,\dot{\theta}$

$\dot{y}=\dot{r}\sin\theta+r\cos\theta\,\dot{\theta}$

$\Rightarrow \dot{x}=r(1-r)\cos\theta-r\sin\theta\sin^2(\theta/2) = x(1-\sqrt{x^2+y^2})-y\sin^2(\frac{1}{2}\tan^{-1}(\frac{y}{x}))$

$\dot{y}=r(1-r)\sin\theta+r\cos\theta\sin^2(\theta/2) = y(1-\sqrt{x^2+y^2})+x\sin^2(\frac{1}{2}\tan^{-1}(\frac{y}{x}))$

- In this case $r=1$ is a homoclinic orbit
- The ray $0\leq x\leq 1$ is a heteroclinic orbit connecting $(0,0)$ to $(1,0)$
- The fixed point $(1,0)$ is an attracting but not Lyapunov stable fixed point.