Lecture 16: Conservative Systems

Conservative Systems:

Inertial systems of the form
$$\ddot{x} = F(x), \quad x \in \mathbb{R}.$$
Can be written as a system
$$\dot{x} = v$$
$$\dot{v} = F(x).$$
A first integral can be found as follows:
$$\dot{x}\ddot{x} = \dot{x}F(x)$$
$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{x}^2) = \frac{d}{dt}V(x),$$
where $V(x) = -\int_{x_0}^{x} F(x)\,dx$.
$$\rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{dV}{dx}\right) = 0$$

For any solution curve ~~the quantity~~ the quantity $E(x,v)$ defined by
$$\boxed{\frac{v^2}{2} + V(x) = E \Rightarrow v = \pm\sqrt{2(E - V(x))}}$$
is constant along the solution curve.

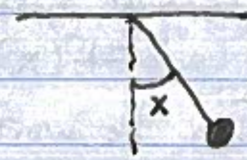
Theorem - A conservative system cannot have any attractors or repellors.

~~proof~~:

Suppose there exists (x^*, v^*) that is an attracting fixed point with basin of attraction A. Then, for all $(x_1,y_1),(x_2,y_2) \in A$, $E(x_1,y_1) = E(x_2,y_2)$ since
$$E(x_1,v_1) = \lim_{t\to\infty} E(x_1(t), v_1(t)) = E(x^*, v^*) = \lim_{t\to\infty} E(x_2(t), v_2(t)) = E(x_2, v_2).$$

Therefore, E must be constant in entire basin of attraction which we preclu[de]
by definition.

Example:



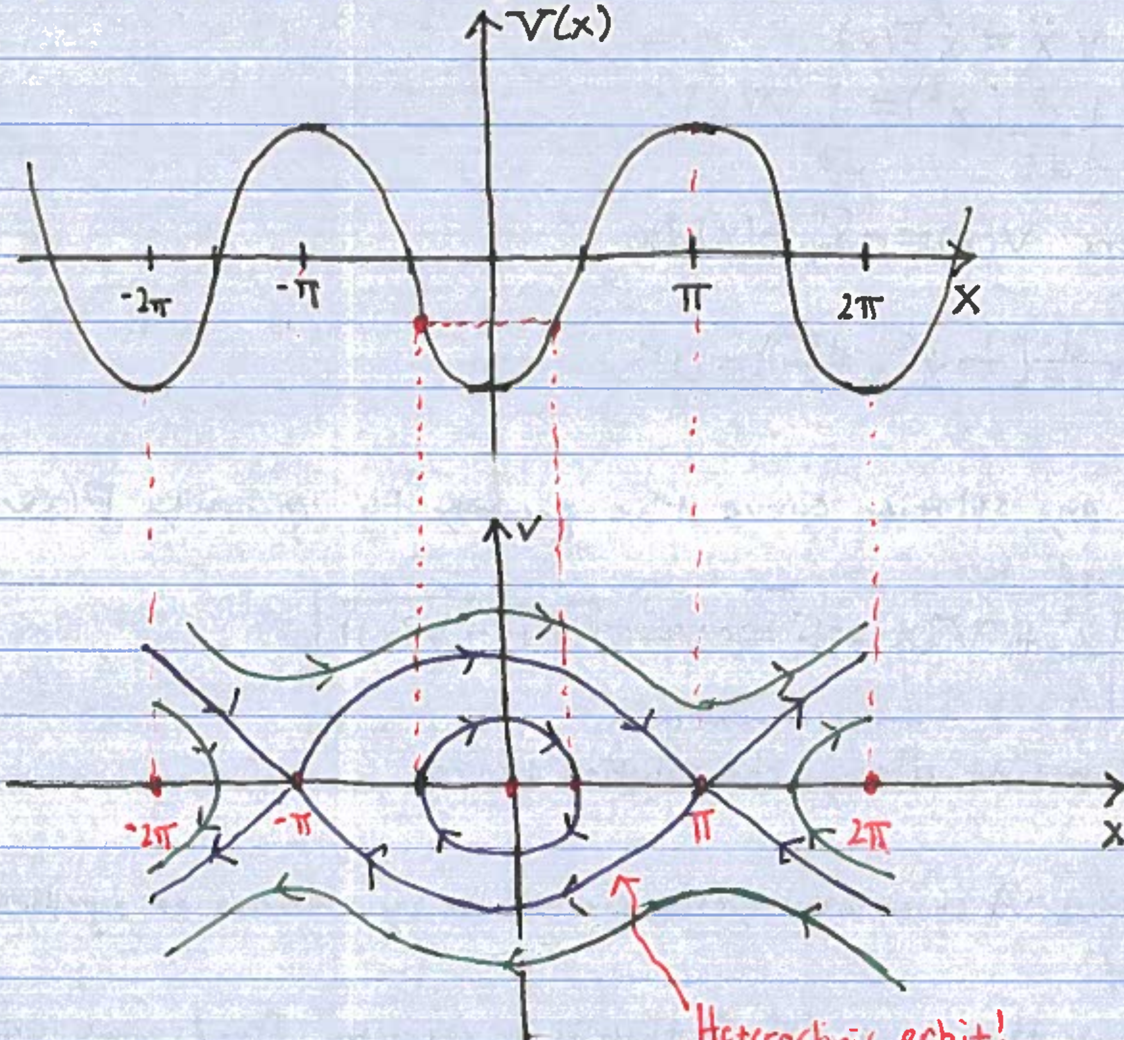
$\ddot{x} + \sin(x) = 0$ (Equation of pendulum)

$\Rightarrow \ddot{x} = -\sin(x)$

$V(x) = -\int -\sin(x)\,dx = -\cos(x)$

$\Rightarrow E = \frac{1}{2}v^2 - \cos(x) = \frac{1}{2}v_0^2 - \cos(x_0)$

$\Rightarrow v = \pm\sqrt{2(\cos(x) - \cos(x_0)) + v_0^2}$



Equation for heteroclin orbit is

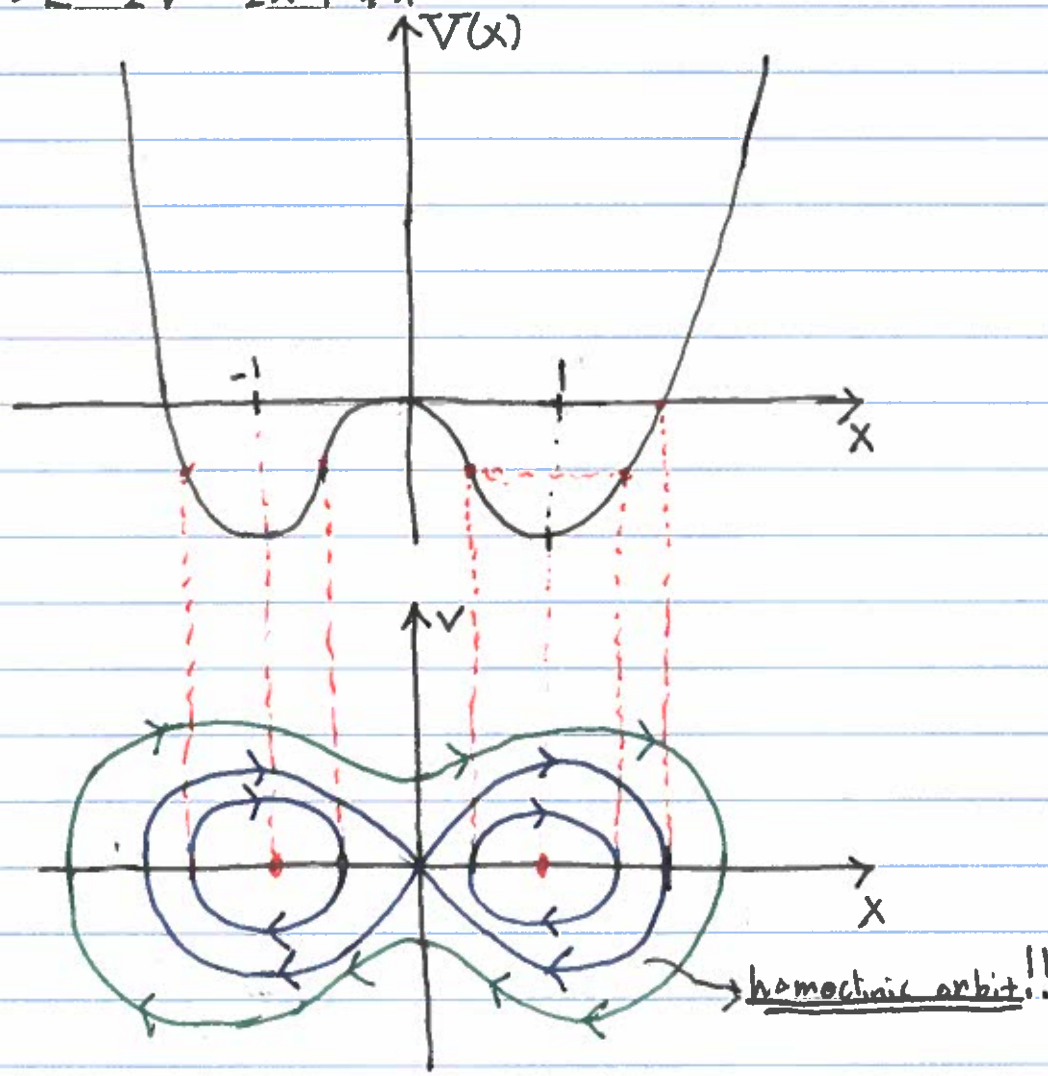
$$v = \pm\sqrt{2(\cos(x) - \cos(-\pi)) + 0^2}$$
$$= \pm\sqrt{2(\cos(x) + 1)}$$

Example:

$$\dot{x} = x - x^3$$

$$V(x) = -\int (x - x^3)\,dx = \frac{x^4}{4} - \frac{x^2}{2}$$

$$\Rightarrow E = \tfrac{1}{2}v^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4$$



Equation for homoclinic orbit satisfies

$$v = \pm\sqrt{2(V(x) - V(0)) + 0^2}$$

$$= \pm\sqrt{\tfrac{1}{2}x^4 - x^2}$$

$$= \pm x\sqrt{\tfrac{1}{2}x^2 - 1}.$$