Lecture 18: Eliminating Limit Cycles

Gradient Systems

$\dot{\vec{x}} = -\nabla V$, where $V : \mathbb{R}^2 \to \mathbb{R}$

Lemma - Gradient systems cannot have periodic orbits.

proof:

Let $x(t)$ be a periodic orbit with period T. Then

$$\frac{d}{dt} V(\vec{x}(t)) = \nabla V \cdot \frac{d\vec{x}}{dt} = -|\nabla V|^2 \leq 0$$

and $V(t)$ is strictly decreasing unless $\vec{x}(t) = \vec{x}^*$ is an equilibrium point. Therefore, $V(\vec{x}(0)) > V(\vec{x}(T))$ which is a contradiction.

Example:

$\dot{x} = 2x + \sin(y)$

$\dot{y} = x\cos(y)$

This system cannot have any periodic orbits

- $V_x = 2x + \sin(y)$
- $V_y = x\cos(y)$

$\Rightarrow V = x^2 + x\sin(y)$

How can we check if a system is gradient?

$$V_{xy} = V_{yx}$$

If $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ then

$$f_y = V_{xy} = V_{yx} = g_x$$

$\Rightarrow \cos(y) = \cos(y)$ (True in our case).

Lyapunov function

$$\dot{\vec{x}} = F(\vec{x})$$

A continuously differentiable function $L: \mathbb{R}^2 \to \mathbb{R}$ is called a Lyapunov function if $L(x(t))$ strictly decreases along each solution of $\dot{\vec{x}} = F(\vec{x})$ that is not an equilibrium.

Lemma - If $\dot{\vec{x}} = F(\vec{x})$ admits a Lyapunov function, then it cannot have any periodic orbits.

Example:

$$\ddot{x} + \alpha \dot{x} = g(x), \quad \alpha > 0$$

Let $V(x) = -\int_{x_0}^{x} g(x)\,dx$. Then

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 + \alpha \dot{x}^2 = -\frac{d}{dt}V(x)$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + V(x)\right) = -\alpha \dot{x}^2 < 0$$

The function $L(v, x) = \frac{1}{2}v^2 + V(x)$ is a Lyapunov function.

Summary:

1. $\ddot{x} = -\frac{dV}{dx} \to$ Conservative, $E(x, \dot{x})$ is conserved. } Many periodic sol.
2. $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla V \to$ Gradient system, V decreases along solutions
3. $\ddot{x} + \alpha \dot{x} = -\frac{dV}{dx} \to E(x, \dot{x})$ decreases.

} No periodic solutions.