Lecture 2: Flows on the Line

General Framework:

$$\frac{dx}{dt} = \dot{x} = f(x), \quad x \in \mathbb{R}, \ f: \mathbb{R} \to \mathbb{R} \text{ differentiable}$$

x – position

$f(x)$ – velocity (really f is a vector field)

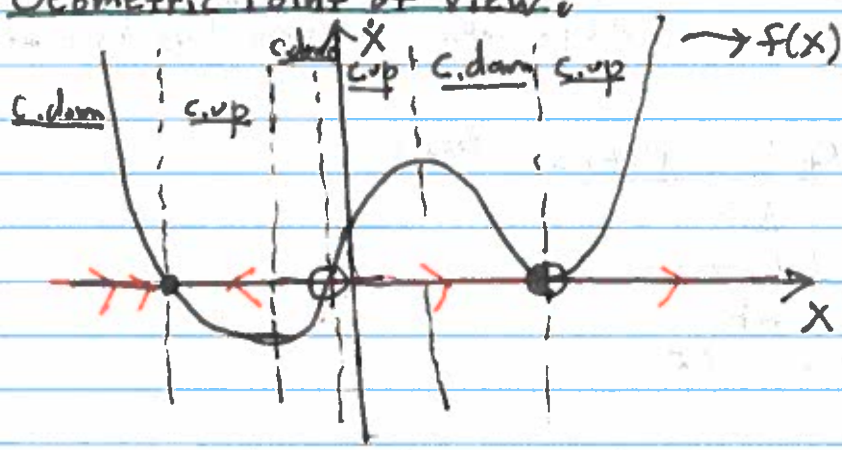
We can try to solve

$$t = \int_{x_0}^{x} f(x)\,dx$$

i) We may not be able to integrate explicitly

ii) It may not be possible to solve for $x(t)$

Geometric Point of View:



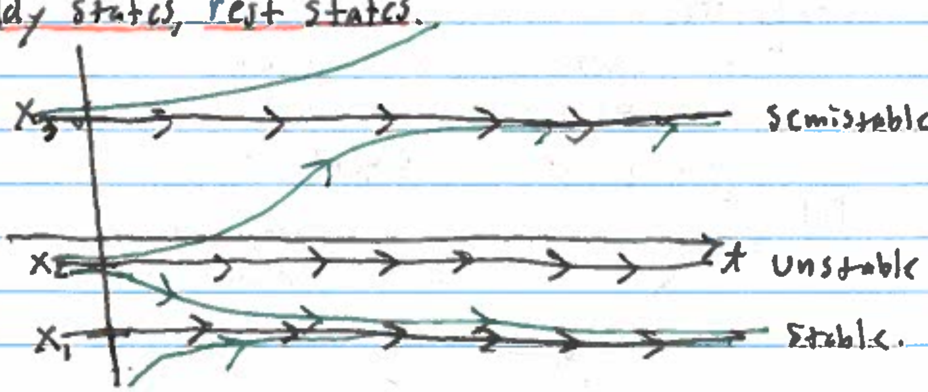
$$\ddot{x} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt} f(x(t)) = \frac{df}{dx}\frac{dx}{dt} = \frac{df}{dx} \cdot f$$

i) $f(x) > 0$, $\frac{dx}{dt} > 0 \Rightarrow x(t)$ increases, moves to the right

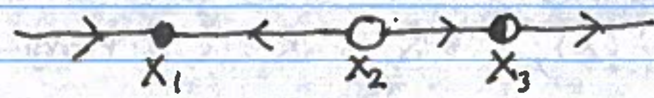
ii) $f(x) < 0$, $\frac{dx}{dt} < 0 \Rightarrow x(t)$ decreases, moves to the left

iii) $f(x) = 0$, $\frac{dx}{dt} = 0$, $x(t) = x_0$ is a solution of $\dot{x} = f(x)$.

*Solutions to $f(x) = 0$ are called fixed points, equilibrium, steady states, rest states.



The phase portrait captures all of this behavior



At each point in $\mathbb{R}$ the function f assigns a tangent vector pointing left or right. Local stability can be determined graphically.

Example (Population Growth)

1. Infinite number of resources, no predators

$$P(t+\Delta t) = P(t) + r\Delta t\, P(t)$$

new population — old population — fraction that reproduce in time Δt.

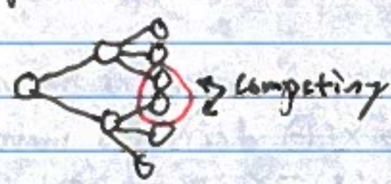
$$\Rightarrow \frac{P(t+\Delta t) - P(t)}{\Delta t} = r P(t)$$

Take $\Delta t \to 0$

$$\Rightarrow \dot{P} = rP$$

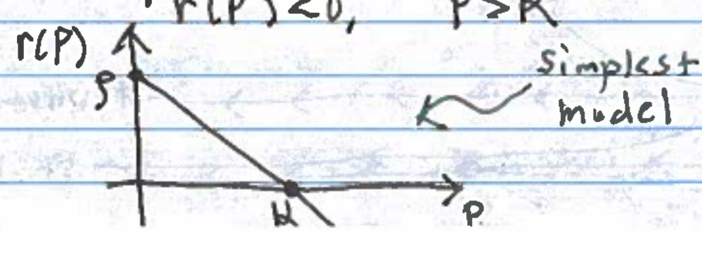
$$\Rightarrow P = P_0 e^{rt}$$

2. A more realistic model must account for finite resources. As population increases it becomes harder to reproduce.



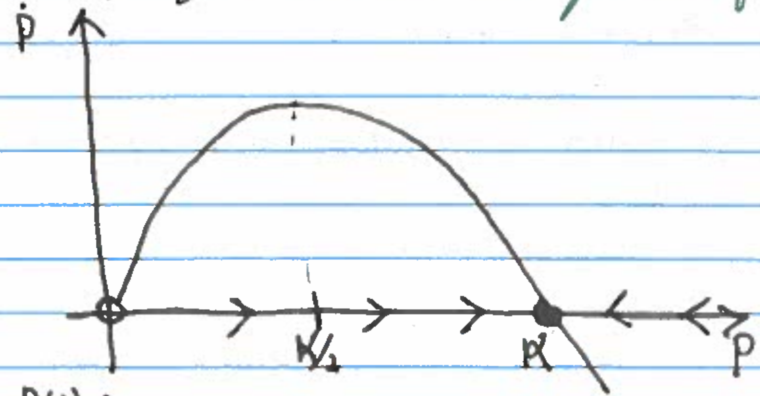
$$P(t+\Delta t) = P(t) + r(P)\Delta t\, P$$

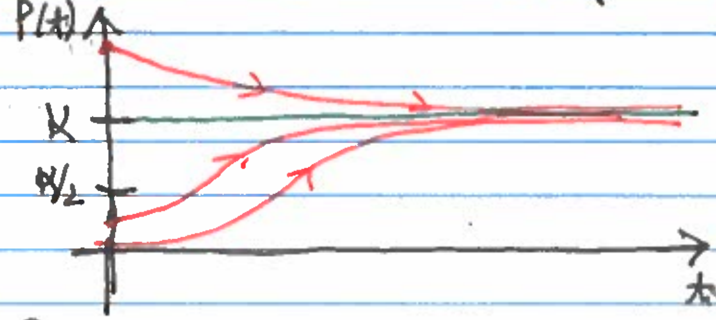
- $r(0) = 0$
- There exists K such that
  - $r(P) > 0, \quad 0 < P < K$
  - $r(P) < 0, \quad P > K$



$$\Rightarrow P(t+\Delta t) = P(t) + \rho(1 - P/K)\Delta t\, P$$

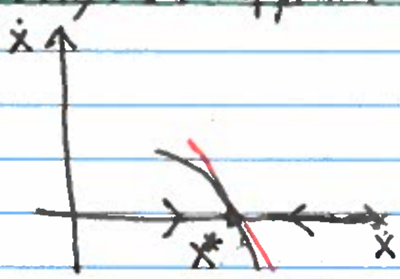
$$\Rightarrow \dot{P} = \rho(1 - P/K)P \rightarrow$$ Logistic equation



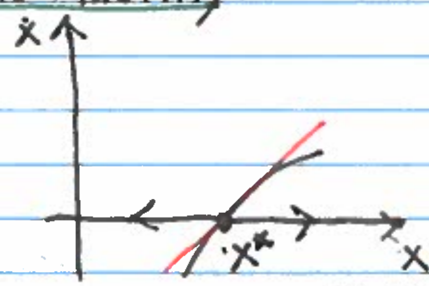


Population goes to carrying capacity.

Analytical Approach to Stability



Stable, $f'(x^*) < 0$



Unstable, $f'(x^*) > 0$

Taylor expand near x^*:

$$f(x) = \underbrace{f(x^*)}_{=0} + f'(x^*)(x - x^*) + \ldots$$

Let $y = x - x^*$ and omit higher order terms:

$$\dot{y} = \dot{x} = f(x) \approx f'(x^*)y$$

$$\Rightarrow y \approx y(0)\exp(f'(x^*)t)$$

initial separation from fixed point

separation grows (shrinks) if $f'(x^*) > 0$ ($f'(x^*) < 0$)

Conclusion:

1. If $f'(x^*) > 0 \Rightarrow x^*$ is unstable
2. If $f'(x^*) < 0 \Rightarrow x^*$ is stable

This process is called linear stability analysis and is a fundamental tool.