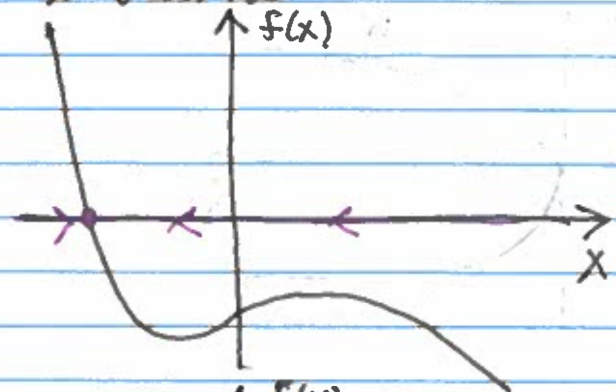
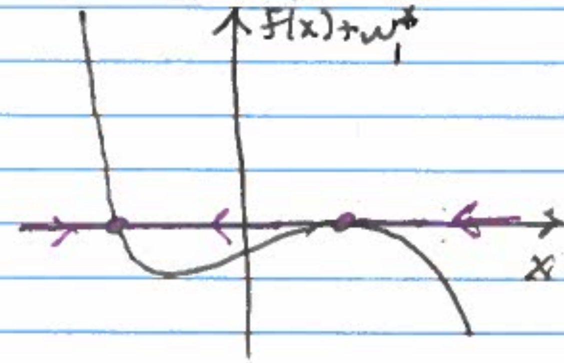
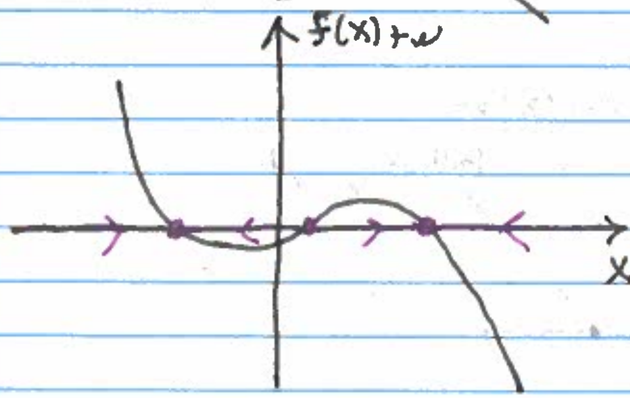
Lecture 6: Saddle Node and Pitchfork Bifurcations

Example:

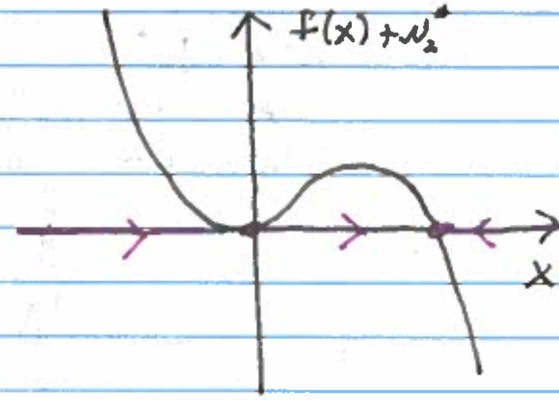
$\dot{x} = f(x) + \nu$



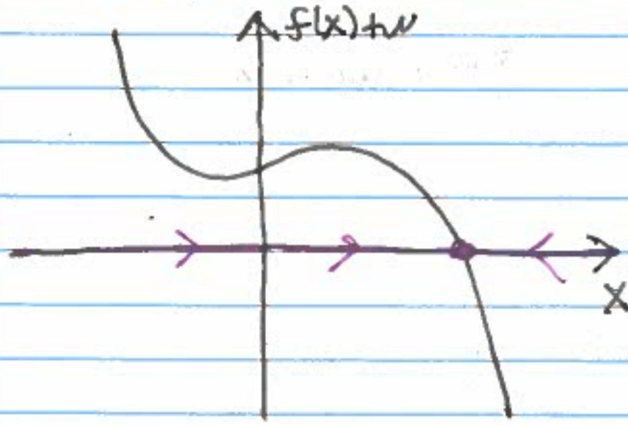




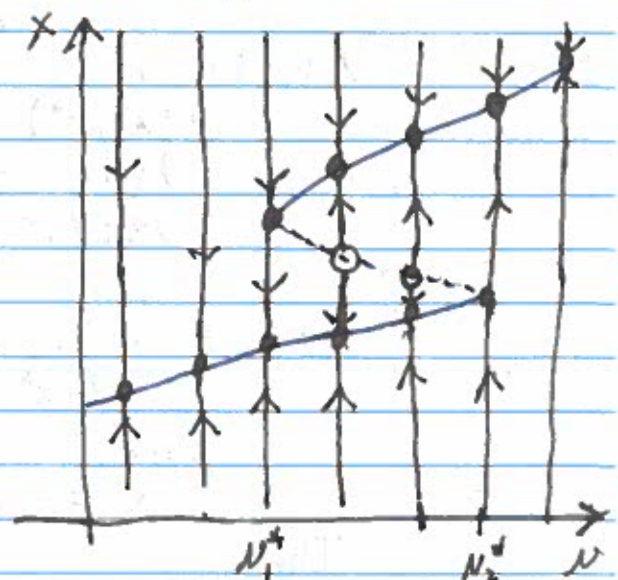
$\nu_1^* < \nu < \nu_2^*$



$\nu = \nu_2^*$



$\nu > \nu_2^*$



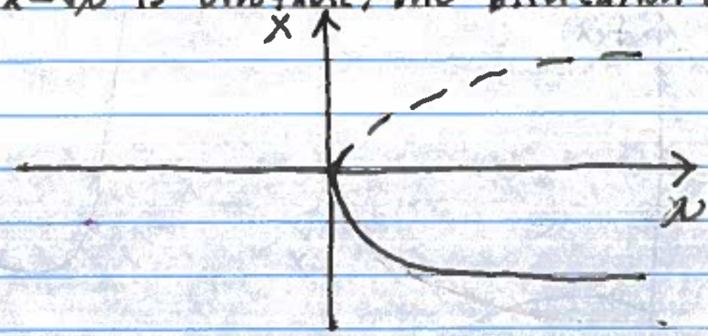
Saddle node bifurcations

Bifurcation points are the values of ν where fixed point changes stability or dissapears.

Example

$\dot{x} = x^2 - \mu$

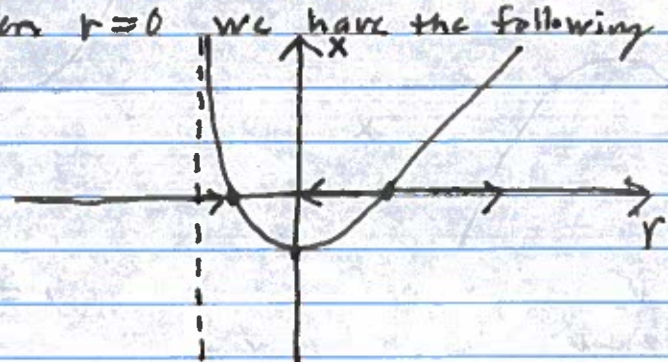
Fixed points are $x = \pm\sqrt{\mu}$. Since $\lim_{x\to\infty} x^2 - \mu = \infty$ it follows that $x = \sqrt{\mu}$ is unstable. The bifurcation diagram is therefore:



Example:

$\dot{x} = r + x - \ln(2+x) = f(x; r)$

When $r = 0$ we have the following plot



At bifurcation point (x^*, r^*) we have

$$f(x^*, r^*) = 0$$

$$\left.\frac{df}{dx}\right|_{x^*, r^*} = 0$$

$$\Rightarrow 1 - \frac{1}{2+x^*} = 0$$

$$2 + x^* = 1$$

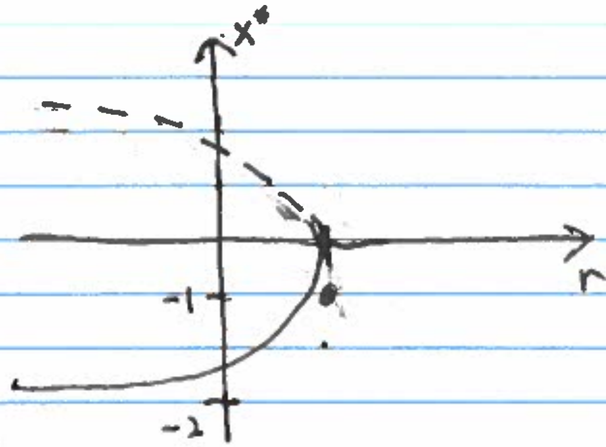
$$x^* = -1$$

Therefore,

$$r^* - 1 - \ln(1) = 0$$

$$\Rightarrow r^* = 1$$

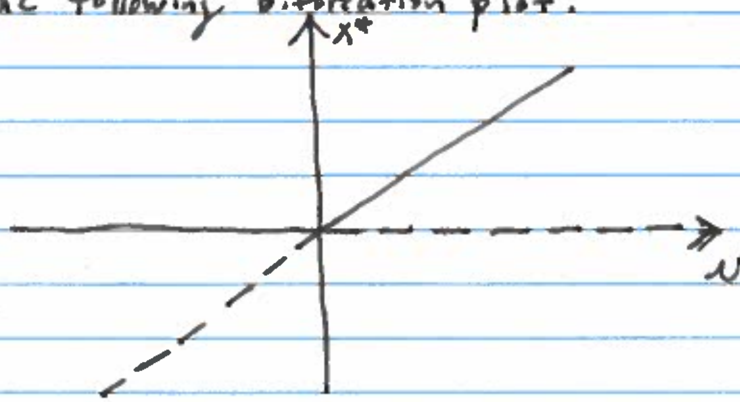
and thus the bifurcation diagram is given by



Example:

$$\dot{x} = \nu x - x^2$$

Fixed points $x^* = 0$, $x^* = \nu$. Since $\lim_{x \to \pm\infty} \nu x - x^2 = -\infty$ we obtain the following bifurcation plot:

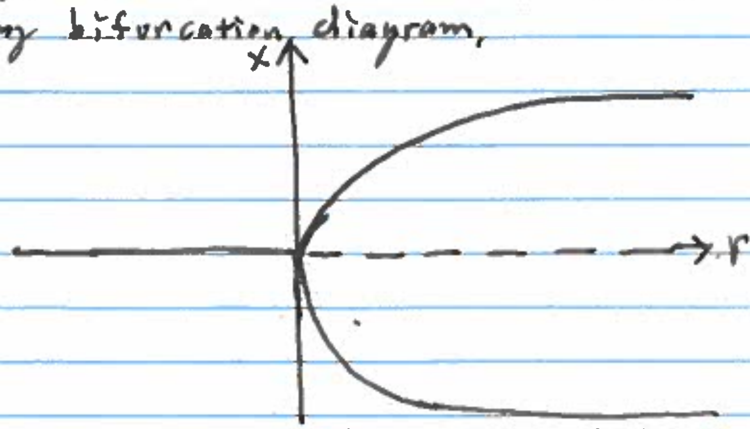


This is a transcritical bifurcation.

Example:

$$\dot{x} = \nu x - x^3$$

Fixed points are $x^* = 0$, $x^* = \pm\sqrt{\nu}$. Since $\lim_{x \to \infty} \nu x - x^3 = -\infty$ the right-most fixed point is stable. Thus, we have the following bifurcation diagram,



This is a supercritical pitchfork bifurcation at $r = 0$