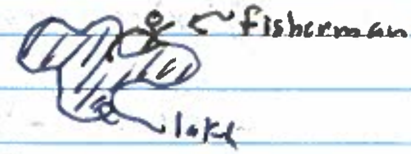
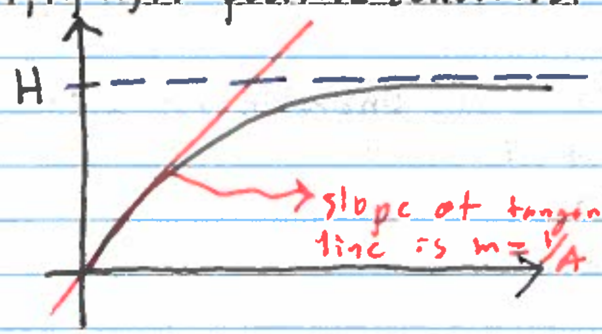
Lecture 7: Model of Fishing in a Lake:

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N} = rN\left(1-\frac{N}{K}\right) - g(N)$$

N - population of fish

r, K, H, A - positive constants.





Max harvesting rate for saturated lake.

$\Rightarrow$ A is a measure of survivability of fish.

Analysis:

$x = N/K$

$\tau = rt$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{\dot{N}}{Kr} = \frac{N}{K}\left(1-\frac{N}{K}\right) - \frac{H}{Kr}\frac{N}{A+N}$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - \frac{H}{r}\frac{x}{A+Kx} = x(1-x) - \frac{H}{rK}\frac{x}{A/K+x}$$

$$\Rightarrow \boxed{\frac{dx}{d\tau} = x(1-x) - \frac{hx}{a+x} = f(x;h,a)}$$

$h = H/rK$

$a = A/K$

Fixed points satisfy

$$x^* = 0, \quad x^* = \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2} = \frac{(1-a) \pm \sqrt{(1+a)^2 - 4h}}{2}$$

Observations:

- $\lim\limits_{x\to\infty} x(1-x) - \frac{hx}{a+x} = -\infty$

  which implies the rightmost fixed point is stable.

- $f'(x) = 1 - 2x - \frac{[(a+x)h - hx]}{(a+x)^2}$

  $\Rightarrow f'(0) = 1 - h/a$

  $\Rightarrow$ 0 is stable if $h > a$, unstable if $h < a$.

- Other roots exist if

  $(a-1)^2 + 4(a-h) > 0$

  $\Rightarrow h < \frac{(a-1)^2}{4} + a$

Case 1:

$h < a < \frac{(a-1)^2}{4} + a$

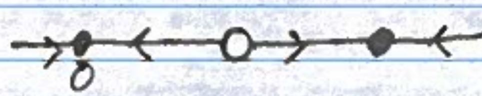
$\Rightarrow$ 

(Robust suvival of fish).

Case 2:

$a < h < \frac{(a-1)^2}{4} + a$

Three fixed points



(Species endangered: Small changes could lead to extinction)

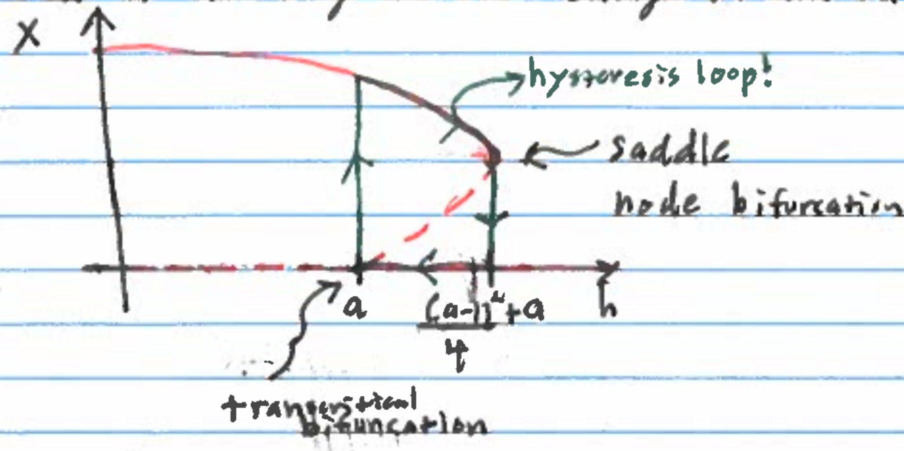
Case 3:

$h > \frac{(a-1)^2}{4} + a$

$\rightsquigarrow$ extinction of fish.

Bifurcation Diagram -

Fix a assuming we can't change A and K.



* Hysteresis: If $h > \frac{(a-1)^2}{4} + a$ the fish population suddenly crashes. This is an example of a tipping point. To rescue the fish population h must be decreased below a!

Phase Diagram -