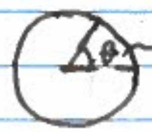
Lecture 9: Flows on a Circle and Fireflies

To fully describe a differential equation
$$\dot{x} = f(x)$$
one must also define the space the solution curve lives on.

Examples:

1. $\mathbb{R}$ - position of a car on a strait track, box on a conveyor belt.

2. $\mathbb{R}^+$ - population growth.
3. S^1 (unit circle) - motion on a circular track, angles, periodic motion.



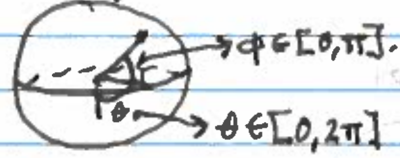
→ θ labels a point

$\theta = 0, 2\pi$ are two labels for the same point.

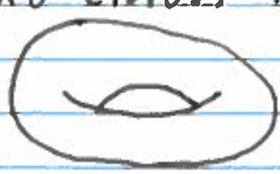
- $\dot{\theta} = \theta^2$ cannot be a vector field on S^1

A vector field on S^1 must satisfy
$$\dot{\theta} = f(\theta + 2n\pi)$$
for all $n \in \mathbb{Z}$.

4. S^2 (unit sphere) - motion on earth



5. $S^1 \times S^1$ (torus) - Two angles



→ two runners on a track.

6. $S^1 \times S^1 \times \mathbb{R}^2$ - automobile

Example:

$\dot{\phi} = \Omega$ entrainment frequency

$\dot{\theta} = \omega + A\sin(\phi - \theta)$

natural frequency — frequency update.

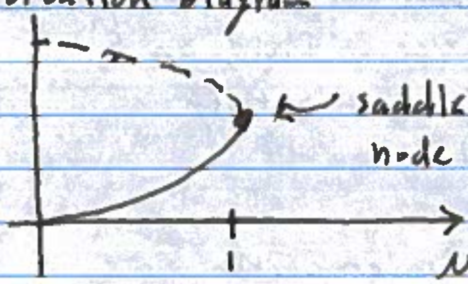
Let $\varphi = \phi - \theta$, rescale $\tau = At$

$$\Rightarrow \frac{d\varphi}{d\tau} = \frac{d\varphi}{dt}\frac{dt}{d\tau} = \frac{1}{A}\dot{\varphi} = \frac{\Omega - \omega}{A} - \sin(\varphi)$$

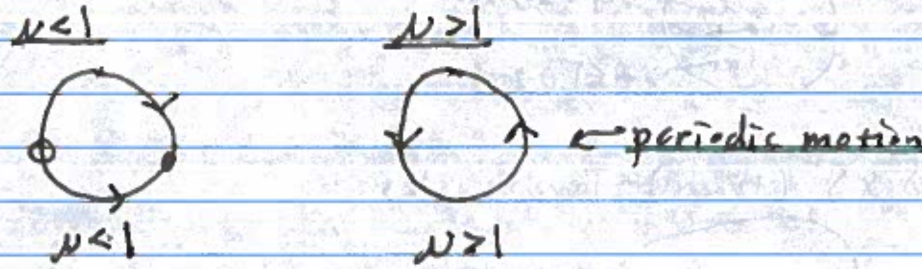
$$\Rightarrow \frac{d\varphi}{d\tau} = \nu - \sin(\varphi), \quad \nu = \frac{\Omega - \omega}{A}$$

Fixed points satisfy $\sin(\varphi) = \nu$

Bifurcation Diagram



← saddle node bifurcation at $\nu = 1$.



We get entrainment if

$$\omega - A \le \Omega \le \omega + A$$

The interval is the range of entrainment.

If $\nu > 1$, the period of the phase drift is:

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\phi}\, d\phi = \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - A\sin(\phi)}\, d\phi = \frac{2\pi}{\sqrt{(\Omega - \omega)^2 - A^2}}.$$