MTH 381
Homework #8

Due Date: December 02, 2022

1 Theory Problems

1. Prove that $f \sim g$ if and only if $f = g$ pointwise-a.e. defines an equivalence relation on the space of all measurable functions.

2. Let $I \subset \mathbb{R}$ be an open interval, and $f : \mathbb{R} \times I \mapsto [-\infty, \infty]$ be a Lebesgue measurable function such that

   (a) $f(\cdot, t) \in L^1(\mathbb{R})$ for each $t \in I$.

   (b) $f(x, \cdot)$ is differentiable in I for each $x \in \mathbb{R}$.

   (c) there is a function $g \in L^1(\mathbb{R})$ such that
   $$\left| \frac{\partial f}{\partial t}(x,t) \right| \leq |g(x)|.$$

   Use the dominated convergence theorem to prove that
   $$\varphi(t) = \int_{\mathbb{R}} f(x,t)\, d\mu$$
   is a differentiable function of t in I and
   $$\frac{d\varphi}{dt} = \int_{\mathbb{R}} \frac{\partial f}{\partial t}(x,t), d\mu.$$

2 Applied Problems

1. Give an example of a sequence of simple functions ϕ_n on $[0,1]$ for which
   $$\lim_{n\to\infty} \int_0^1 \phi_n(x) = 0$$
   while
   $$\lim_{n\to\infty} \phi_n(x)$$
   exists for no $x \in [0,1]$. **Hint:** You might not be able to find an explicit formula but you can draw pictures.

2. For $f \in L^1(\mathbb{R})$ find
   $$\lim_{A\to\infty} \int_{-\infty}^{\infty} \frac{f(x)}{A^2 + x^2} dx.$$

3. The convolution of two real value functions f and $g \in^{L^1}(\mathbb{R})$ is defined by

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Prove that

(a) $\|f \star g\|_{L^1} \leq \|g\|_{L^1}\|f\|_{L^1}$.

(b) $\|f \star g\|_{L^\infty} \leq \|g\|_{L^1}\|f\|_{L^\infty}$.

4. Consider the function

$$E(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

(a) Using the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, show that for all $t > 0$,

$$\int_{-\infty}^{\infty} E(t,x)dx = 1.$$

(b) Show that $E(t,x)$ solves the following partial differential equation

$$\frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2}.$$

(c) Show that for $f \in C^1(\mathbb{R})$:

$$\lim_{t \to 0} (E \star f)(t,x) = f(x).$$

(d) Show that $u(x,t) = (E \star f)(t,x)$ satisfies the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the initial condition $u(0,x) = f(x)$. **Hint:** Don't worry about differentiating under the integral sign.

5. Let f be a non-negative measurable function. Prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \frac{f^n(x)}{1+f^n(x)} d\mu(x) = \mu(E) + \frac{1}{2}\mu(F),$$

where $E = \{x \in X : f(x) > 1\}$, $F = \{x \in X : f(x) = 1\}$ and μ is the Lebesgue measure.