Lecture 3: Riemann Integral

- Partition: Suppose $a, b \in \mathbb{R}$ with $a<b$. A partition of $[a,b]$ is a list of the form $x_0, \dots, x_n$ where
$$a = x_0 < x_1 < \cdots < x_n = b.$$

- Lower and Upper Riemann sums

For a partition P:

- $L(f, P, [a,b]) = \sum (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$ (Lower sum)
- $U(f, P, [a,b]) = \sum_1 (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f$ (Upper sum)

- $L(f, [a,b]) = \sup_P L(f, P, [a,b])$
- $U(f, [a,b]) = \inf_P U(f, P, [a,b])$

- Riemann integrable

A bounded function on a closed bounded interval I. is called Riemann integrable on I if
$$L(f,[a,b]) = U(f,[a,b]) = \int_a^b f(x)\,dx.$$

Example:

Define the Dirichlet function $D: [0,1] \to \mathbb{R}$ by
$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{o.w.} \end{cases}$$

For any interval $[a,b] \subset [0,1]$
$$\inf_{[a,b]} D(x) = 0, \quad \sup_{[a,b]} D(x) = 1$$
$$\Rightarrow L(D, [0,1]) = 0$$
$$U(D, [0,1]) = 1$$

Assignment for students:

1. Example 1.4, 1.10 Molly
2. 1.5, Ashley
3. 1.6, Malindi
4. 1.8. Molly
5. 1.11 Ashley
6. 1.13 Malindi
7. 1.15 Molly
8. 1.16 Ashley
9. 1.17 Malindi