MTH 317/617
Homework #10

Due Date: December 8, 2023

1 Problems for Everyone

1. Evaluate the following integrals using contour integration. You must show and justify your steps to receive full credit.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{1+x^8}dx.$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4-4x^2+5}.$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}dx$, where $a > b > 0$.

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)(x^2+4)}dx$

   (e) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^4+1}dx$

   (f) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+6x+10}dx$

   (g) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(x+a)^2+b^2}$, where $a, b \in \mathbb{R}$.

Homework #10

#1

Evaluate the following integrals using contour integration.

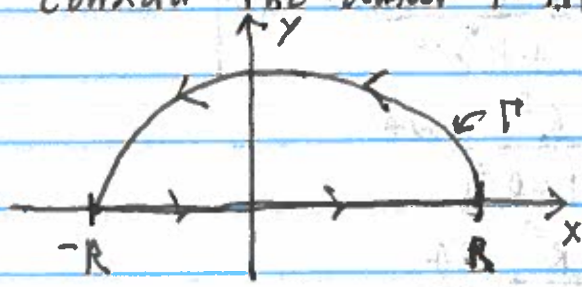
(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\,dx$

Solution:

Let $f:\mathbb{C}\to\mathbb{C}$ be given by

$$f(z)=\frac{z^4}{1+z^8}$$

and consider the contour Γ illustrated below



The singularities of $f(z)$ inside Γ are given by

$$z^* = e^{i\pi/8},\ e^{3i\pi/8},\ e^{5i\pi/8},\ e^{7i\pi/8}.$$

The residues at these points are given by

$$-\operatorname{Res}(f;e^{i\pi/8}) = \left.\frac{z^4}{8z^7}\right|_{z=e^{i\pi/8}} = \left.\frac{1}{8z^3}\right|_{z=e^{i\pi/8}} = \frac{e^{-3i\pi/8}}{8} = \frac{e^{13i\pi/8}}{8}$$

$$-\operatorname{Res}(f;e^{3i\pi/8}) = \frac{e^{-9i\pi/8}}{8} = \frac{e^{7i\pi/8}}{8}$$

$$-\operatorname{Res}(f;e^{5i\pi/8}) = \frac{e^{-15i\pi/8}}{8} = \frac{e^{i\pi/8}}{8}$$

$$-\operatorname{Res}(f;e^{7i\pi/8}) = \frac{e^{-21i\pi/8}}{8} = \frac{e^{-5i\pi/8}}{8} = \frac{e^{11i\pi/8}}{8}$$

Therefore,

$$-\operatorname{Res}(f; e^{i\pi/8}) = \frac{1}{8}\left(\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)\right) = \frac{1}{8}\left(\cos\left(\frac{3\pi}{8}\right) - i\sin\left(\frac{3\pi}{8}\right)\right)$$

$$-\operatorname{Res}(f; e^{3i\pi/8}) = \frac{1}{8}\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right) = \frac{1}{8}\left(-\cos\left(\frac{\pi}{8}\right) + i\sin(\pi/8)\right)$$

$$-\operatorname{Res}(f; e^{5i\pi/8}) = \frac{1}{8}\left(\cos(\pi/8) + i\sin(\pi/8)\right)$$

$$-\operatorname{Res}(f; e^{7i\pi/8}) = \frac{1}{8}\left(\cos(11\pi/8) + i\sin(11\pi/8)\right) = \frac{1}{8}\left(-\cos\left(\frac{3\pi}{8}\right) - i\sin\left(\frac{3\pi}{8}\right)\right).$$

Consequently,

$$\int_{\Gamma} \frac{z^4}{1+z^8}\,dz = \frac{2\pi i}{8}\left(2\sin(\pi/8)i - 2i\sin(3\pi/8)\right)$$
$$= \frac{\pi}{2}\left(\sin(3\pi/8) - \sin(\pi/8)\right).$$

Furthermore, along the radial part of the contour Γ_R we have that

$$\left|\int_{\Gamma_R} \frac{z^4}{1+z^8}\,dz\right| = \left|\int_0^{\pi} \frac{R^4 e^{4it}}{1+R^8 e^{it}} \cdot Re^{it}\,dt\right|$$
$$\leq \int_0^{\pi} \frac{R^5}{|1+R^8|}\,dt$$
$$\leq \int_0^{\pi} \frac{R^5}{R^8-1}\,dt$$
$$= \frac{\pi R^5}{R^8-1},$$

assuming $R>1$. Therefore, by the squeeze theorem

$$\frac{\pi}{2}\left(\sin(3\pi/8) - \sin(\pi/8)\right) = \lim_{R\to\infty}\int_{\Gamma} \frac{z^4}{1+z^8}\,dz = \int_{-\infty}^{\infty} \frac{x^4}{1+x^8}\,dx.$$

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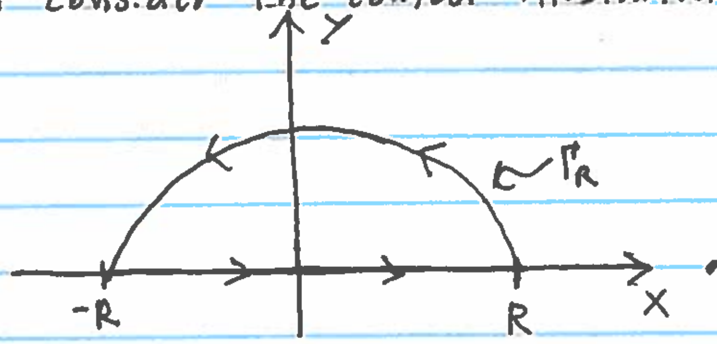
(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4-4x^2+5}\,dx$

<u>Solution:</u>

Let $f:\mathbb{C}\to\mathbb{C}$ be given by

$$f(z)=\frac{z^2}{z^4-4z^2+5}$$

and consider the contour illustrated below



The singularities of $f(z)$ satisfy

$$z^4-4z^2+5$$

$$\Rightarrow (z^*)^2=\frac{4\pm\sqrt{16-20}}{2}$$

$$=2\pm i$$

Therefore, the singularities inside Γ are given by

$$z^*=\sqrt{2+i},\ -\sqrt{2-i}.$$

Consequently,

$$\int_\Gamma \frac{z^2}{z^4-4z^2+5}\,dz = 2\pi i\left(\left.\frac{z^2}{4z^3-8z}\right|_{\sqrt{2+i}} + \left.\frac{z^2}{4z^3-8z}\right|_{-\sqrt{2+i}}\right)$$

$$=\frac{2\pi i}{4}\left(\left.\frac{z}{z^2-2}\right|_{\sqrt{2+i}} + \left.\frac{z}{z^2-2}\right|_{-\sqrt{2-i}}\right)$$

$$=\frac{\pi i}{2}\left(\frac{\sqrt{2+i}}{i}+\frac{\sqrt{2-i}}{-i}\right)$$

$$\Rightarrow \int_{\Gamma} \frac{z^2}{z^4-4z^2+5}\,dz = \frac{\pi}{2}\left(\sqrt{2+i}+\sqrt{2-i}\right).$$

Furthermore,

$$\left|\int_{\Gamma_R} \frac{z^2}{z^4-4z^2+5}\,dz\right| = \left|\int_0^{\pi} \frac{R^2e^{2i\theta}\cdot iRe^{i\theta}}{R^4e^{4i\theta}-4R^2e^{2i\theta}+5}\,d\theta\right|$$

$$\leq \int_0^{\pi} \frac{R^3}{\left|R^4-|4R^2e^{2i\theta}-5|\right|}\,d\theta$$

$$\leq \int_0^{\pi} \frac{R^3}{R^4-4R^2+5}\,d\theta = \frac{\pi R^3}{R^4-4R^2+5},$$

for sufficiently large R. Therefore, by the squeeze theorem

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{z^2}{z^4-4z^2+5}\,dz = 0.$$

Finally,

$$\frac{\pi}{2}\left(\sqrt{2+i}+\sqrt{2-i}\right) = \lim_{R\to\infty} \int_{\Gamma} \frac{z^2}{z^4-4z^2+5}\,dz = \frac{\pi}{2}\left(\sqrt{2+i}+\sqrt{2-i}\right).$$

$\blacksquare$

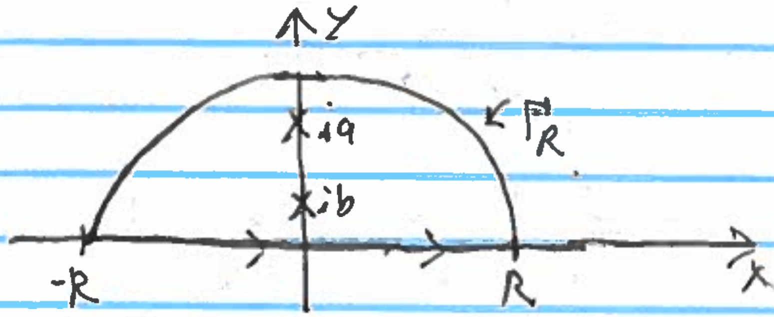
(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$, $a>b>0$.

Solution:

Let $f: \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}.$$

and consider the contour draw below:



It follows that

$$\int_{\Gamma} \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz = 2\pi i \left(\frac{1}{4z^3+2a^2z+2b^2z}\bigg|_{ia} + \frac{1}{4z^3+2a^2z+2b^2z}\bigg|_{ib} \right)$$

$$= 2\pi i \left(\frac{1}{-4ia^3+2ia^3+2b^2ia} + \frac{1}{-4ib^3+2a^2ib+2b^3} \right)$$

$$= \pi \left(\frac{1}{b^2a-a^3} + \frac{1}{a^2b-b^3} \right)$$

$$= \frac{\pi}{a(b^2-a^2)b(a^2-b^2)} \left(a^2b-b^3+b^2a-a^3 \right)$$

$$= \frac{\pi}{ba(b-a)(a-b)(b+a)^2} \left(b(a^2-b^2)+a(b^2-a^2) \right)$$

$$= \frac{\pi(a^2-b^2)\cancel{(b-a)}}{ba\cancel{(b-a)}(a-b)(b+a)^2}$$

$$= \frac{\pi}{ab(a+b)}.$$

Furthermore,

$$\left| \int_{\Gamma_R} \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz \right| \le \int_0^{\pi} \frac{R}{(R^2-a^2)(R^2-b^2)}\,d\theta = \frac{\pi R}{(R^2-a^2)(R^2-b^2)},$$

for $R > a$. Therefore, by the squeeze theorem

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{1}{(z^2+a^2)(z^2+b^2)}\,dz=\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)(x^2+b^2)}\,dx=\frac{\pi}{ab(a+b)}.$$

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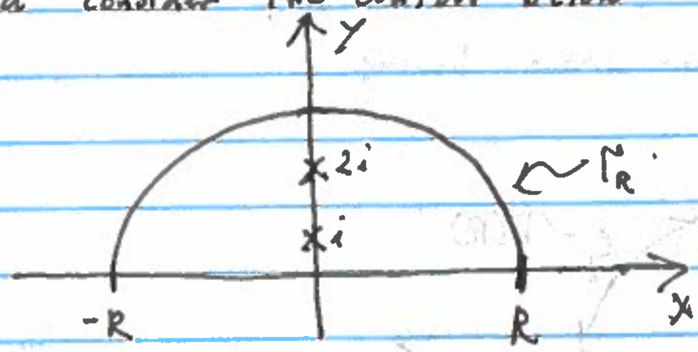
(d) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(x)}{(x^2+1)(x^2+4)}\,dx.$

Solution:

Let $f:\mathbb{C}\to\mathbb{C}$ be defined by

$$f(z)=\frac{e^{iz}}{(z^2+1)(z^2+4)}.$$

and consider the contour below:



Therefore,

$$\int_{\Gamma}\frac{e^{iz}}{(z^2+1)(z^2+4)}\,dz=2\pi i\left(\frac{e^{iz}}{(4z^3+10z)}\Bigg|_{i}+\frac{e^{iz}}{(4z^3+10z)}\Bigg|_{2i}\right)$$

$$=2\pi i\left(\frac{e^{-1}}{6i}+\frac{e^{-2}}{-12i}\right)$$

$$=\pi\left(\frac{1}{3e}-\frac{1}{4e^2}\right)$$

Furthermore,

$$\left|\int_{\Gamma_R}\frac{e^{iz}}{(z^2+1)(z^2+4)}\,dz\right|\le\int_0^{\pi}\frac{Re^{-R\sin t}}{(R^2-1)(R^2-4)}\,dt\le\frac{\pi R}{(R^2-1)(R^2-4)}.$$

Therefore, by squeeze theorem

$$\pi\left(\frac{1}{3e}-\frac{1}{4e^2}\right) = Re\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+1)(x^2+4)}\,dx\right) = \int_{-\infty}^{\infty}\frac{\cos(x)}{(x^2+1)(x^2+4)}\,dx$$

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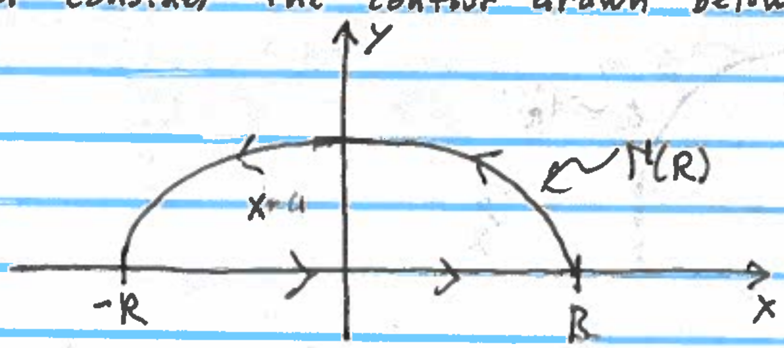
(g) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(x)}{(x+a)^2+b^2}\,dx$

<u>Solution:</u>

Let $f:\mathbb{C}\to\mathbb{C}$ be defined by

$$f(z)=\frac{e^{iz}}{(z+a)^2+b^2}.$$

and consider the contour drawn below



Therefore,

$$\int_{\Gamma}\frac{e^{iz}}{(z+a)^2+b^2}\,dz = \frac{e^{iz}}{2(z+a)}\Bigg|_{z=bi-a} 2\pi i = \frac{e^{ia}e^{-b}}{\cancel{2}\cdot b\cancel{i}}\cdot\cancel{2}\pi\cancel{i} = \frac{\pi e^{-b}}{b}e^{ia}$$

Therefore, showing the radial part goes to 0 we have that

$$\frac{\pi e^{-b}}{b}\cos(a) = Real\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{(x+a)^2+b^2}\,dx\right) = \int_{-\infty}^{\infty}\frac{\cos(x)}{(x+a)^2+b^2}\,dx$$

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