MTH 317/617: Homework #1

Due Date: September 08, 2023

1 Problems for Everyone

1. If $z = 1 + 2i$, $w = 2 - i$ and $\zeta = 4 + 3i$, write the following complex expressions in the form $a + bi$ where a and b are real numbers.

   (a) $z + 3w$,

   (b) $-2w + \bar{\zeta}$,

   (c) z^2,

   (d) $w^3 + w$,

   (e) $\mathrm{Im}(\zeta^{-1})$,

   (f) w/z,

   (g) $\zeta^2 + 2\bar{\zeta} + 3$.

2. Solve the following equations for z. Express your answer in the form $z = a + bi$ where a and b are real numbers.

   (a) $z = 1 - zi$,

   (b) $\dfrac{z}{1+z} = 1 + 2i$,

   (c) $(\pi + i)z - 8z^2 = 0$,

   (d) $z^2 + i = 0$.

3. Describe the set of points $z \in \mathbb{C}$ that satisfy each of the following.

   (a) $|z - 1 + 1| = 3$,

   (b) $|z - 1| = |z + 1|$,

   (c) $|z| = \mathrm{Re}(z) + 2$,

   (d) $2 < |z| < 6$,

   (e) $\mathrm{Re}(z/(1 + i)) = 0$.

4. Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following:

   (a) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$,

   (b) $\mathrm{Re}(z) = (z + \overline{z})/2$ and $\mathrm{Im}(z) = -i(z - \overline{z})/2$,

   (c) If k is an integer then $(\overline{z})^k = \overline{(z^k)}$,

   (d) $|z| = 1$ if and only if $1/z = \bar{z}$.

   (e) If $|z| = 1$ and $z \neq 1$, then $\mathrm{Re}\left((1 - z)^{-1}\right) = 1/2$.

5. Find the argument of the following complex numbers and write each in the polar form $z = r(\cos(\theta) + i\sin(\theta))$.
   (a) $-1+i$,
   (b) $1+i\sqrt{3}$,
   (c) $-i$,
   (d) $(2-i)^2$,
   (e) $|4+3i|$,
   (f) $\sqrt{2}/(1+i)$,
   (g) $[(1+i)/\sqrt{2}]^4$,
6. Write the given complex number in the form $a+bi$, where $a, b \in \mathbb{R}$.
   (a) $e^{-i\frac{\pi}{2}}$,
   (b) $\dfrac{e^{1+3\pi i}}{e^{-1+\frac{\pi i}{2}}}$,
   (c) $\dfrac{e^{3i}-e^{-3i}}{2i}$,
   (d) $e^{e^{i}}$.

2 Graduate Problems

1. In this exercise you will prove the Cauchy-Schwarz inequality for complex numbers.
   (a) Let B, C be nonnegative real numbers and suppose that

$$0 \leq B - 2\mathrm{Re}(\bar{\lambda}A) + |\lambda|^2 C$$

   for all $\lambda \in \mathbb{C}$. Prove that $|A|^2 \leq BC$.
   (b) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers. Prove the following inequality:

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{i=1}^{n} |b_j|^2. \qquad (1)$$

   Hint: For all $\lambda \in \mathbb{C}$, we have $0 \leq \sum_{j=1}^{n} |a_j - \lambda b_j|^2$.
   (c) When does equality hold in (1)?
   (d) Use (1) to prove that

$$\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$$

2. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Prove that $|z| \leq |x| + |y|$.

<u>Homework #1</u>

#2. Solve the following equations for z.

(a) $z = 1 - zi$

(b) $z/1+z = 1+2i$

(c) $(\pi + i)z - 8z^2 = 0$

(d) $z^2 + i = 0$

<u>Solution:</u>

(a) $z = 1 - zi$

$\Rightarrow z(1+i) = 1$

$\Rightarrow z = 1/1+i$

$\Rightarrow z = \frac{1}{2}(1-i) = \frac{1}{2} - \frac{i}{2}$

(b) $z/1+z = 1+2i$

$\Rightarrow z = 1+2i + z(1+2i)$

$\Rightarrow -2iz = 1+2i$

$\Rightarrow z = \dfrac{-1}{2i} - 1$

$\Rightarrow z = -1 + \frac{1}{2}i$

(c) $(\pi + i)z - 8z^2 = 0$

$\Rightarrow z(\pi + i - 8z) = 0$

$\Rightarrow z = 0$ or $z = \pi/8 + i/8$

(d) If $z^2 = -i$ and $z = x+iy$ then

$x^2 - y^2 + 2ixy = -i$

$\Rightarrow y = -1/2x, \quad y^2 = x^2$

$\Rightarrow \dfrac{1}{4x^2} = x^2$

$\Rightarrow x = \pm 1/\sqrt{2} \Rightarrow z = 1/\sqrt{2} - i\,1/\sqrt{2}$ or $z = -1/\sqrt{2} + i\,1/\sqrt{2}$.

#3

Describe the set of points $z \in \mathbb{C}$ that satisfy the following.

(a) $|z-1+1|=3$

(b) $|z-1|=|z+1|$

(c) $|z|=Re(z)+2$

(d) $2<|z|<6$

Solution:

(a) $|z|=3$ coorespond s to a circle of radius $\sqrt{3}$ centered at the origin.

(b) If we let $z=x+iy$ then $|z-1|=|z+1|$ coorresponds to the equations

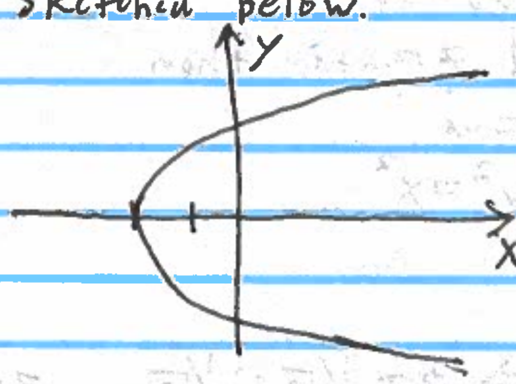
$$(x-1)^2+y^2=(x+1)^2+y^2$$
$$\Rightarrow x^2-2x=x^2+2x$$
$$\Rightarrow x=0.$$

Therefore, this line coorresponds to the imaginary axis.

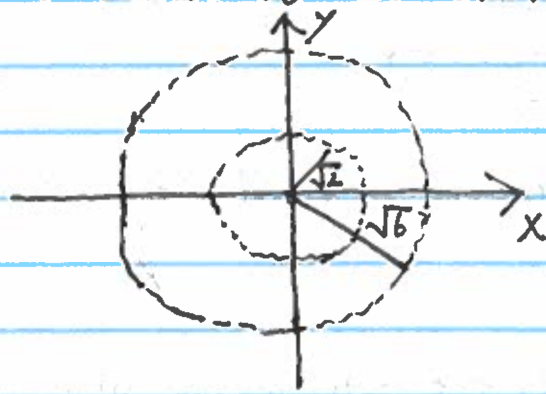
(c) If we let $z=x+iy$ then $|z|=Re(z)+2$ coorresponds to the equation $\sqrt{x^2+y^2}=x+2$ and thus for $x\geq -2$ we have

$$x^2+y^2=x^2+2x+4$$
$$\Rightarrow y^2=2x+4$$

which coorresponds to the graph of the rightward opening parabola sketched below.



(d) The set described by $2<|z|<6$ is an annulus centered at the origin with inner radius $\sqrt{2}$ and outer radius $\sqrt{6}$ as sketched below!



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#4 Let $z\in\mathbb{C}$ and assume $z\neq 0$. Prove the following

(b) $\mathrm{Re}(z)=(z+\bar{z})/2$ and $\mathrm{Im}(z)=-i(z-\bar{z})/2$.

(c) If $k\in\mathbb{Z}$ then $(\bar{z})^k=\overline{(z^k)}$.

(d) $|z|=1$ if and only if $1/z=\bar{z}$

(e) If $|z|=1$ and $z\neq 1$, then $\mathrm{Re}((1-z)^{-1})=1/2$.

Solution:

~~proof~~:

(b) If $z\in\mathbb{C}$ then there exists $x,y\in\mathbb{R}$ such that $z=x+iy$.

Consequently,

$$\frac{z+\bar{z}}{2}=\frac{x+iy+x-iy}{2}=x=\mathrm{Re}(z)$$

$$\frac{-i(z-\bar{z})}{2}=\frac{-i(x+iy-x+iy)}{2}=y=\mathrm{Im}(z)$$

(c) If $k\in\mathbb{Z}$ then by De Moivre's theorem we have:

$$\bar{z}^k=|z|^k(\cos\theta-i\sin\theta)^k=|z|^k(\cos(-\theta)+i\sin(-\theta))^k$$

$$\Rightarrow \bar{z}^k=|z|^k(\cos(-k\theta)+i\sin(-k\theta))=|z|^k(\cos(k\theta)-i\sin(k\theta))$$

$$\Rightarrow \bar{z}^k=\overline{z^k}.$$

(d) If $z \in \mathbb{C}$, then there exists $x, y \in \mathbb{R}$ such that $z = x + iy$. Therefore, $|z| = 1$ if and only if

$$1 = |z|^2 = x^2 + y^2$$
$$= (x+iy)(x-iy)$$
$$= z \cdot \bar{z},$$

Therefor, $|z| = 1$ if and only if $z = 1/\bar{z}$.

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#5. Find the argument of the following complex numbers and write each in the polar form $z = r(\cos\theta + i\sin\theta)$.

(a) $-1 + i$

(c) $-i$

(e) $|4 + 3i|$

(g) $[(1+i)/\sqrt{2}]^4$

Solution:

(a) If $z = -1 + i$ then $\text{Arg}(z) = 3\pi/4$ and $z = \sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$.

(c) If $z = -i$ then $\text{Arg}(z) = -\pi/2$ and $z = \cos(-\pi/2) + i\sin(-\pi/2)$.

(e) If $z = |4 + 3i|$ then $\text{Arg}(z) = 0$ and $z = 5(\cos(0) + i\sin(0))$.

(g) If $z = [(1+i)/\sqrt{2}]^4$ then $z = (e^{i\pi/4})^4 = e^{i\pi}$ and thus $\text{Arg}(z) = \pi$. Therefore, $z = (\cos(\pi) + i\sin(\pi))$.

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#6 Write the given complex number in the form $a+bi$, where $a,b \in \mathbb{R}$.

Solution:

(a) $e^{-i\pi/2} = -i$

(b). $\dfrac{e^{1+3\pi i}}{e^{-1+\pi i/2}} = e^{2+5\pi i/2} = e^2(\cos(5\pi i/2) + i\sin(5\pi i/2)) = ie^2.$

(c) $\dfrac{e^{3i}-e^{-3i}}{2i} = \sin(6).$

(d) $e^{e^i} = e^{(\cos(1)+i\sin(1))} = e^{\cos(1)}(\cos(\sin(1)) + i\sin(\sin(1))).$

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Graduate Problems

#1 In this exercise you will prove the Cauchy-Schwarz inequality for complex numbers.

(a) Let $B, C > 0$ and suppose

$$0 \le B - 2\mathrm{Re}(\bar{\lambda} A) + |\lambda|^2 C$$

for all $\lambda \in \mathbb{C}$. Prove that $|A|^2 \le BC$.

(b) Let $a_1, \dots, a_n$ and $b_1, \dots, b_n \in \mathbb{C}$. Prove the following

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 \qquad (1)$$

(c) When equality in (1) hold?

(d) Use (1) to prove that

$$\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2} \le \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}.$$

Solution:

(a) Suppose for $B, C > 0$ and $A \in \mathbb{C}$ that
$$0 \leq B - 2\,\mathrm{Re}(\bar{\lambda} A) + |\lambda|^2 C$$
for all $\lambda \in \mathbb{C}$. Therefore, if $\lambda = A/C$ we obtain
$$0 \leq B - 2|A|^2/C + |A|^2/C$$
$$\Rightarrow |A|^2 \leq BC.$$

(b). Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n \in \mathbb{C}$. Since
$$0 \leq \sum_{j=1}^{n} |a_j - \lambda b_j|^2 = \sum_{j=1}^{n} |a_j|^2 - 2\,\mathrm{Re} \sum_{j=1}^{n} (\bar{\lambda} a_j \bar{b}_j) + |\lambda|^2 \sum_{j=1}^{n} |b_j|^2$$
it follows from part (a) that
$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right).$$

(c) Equality holds when $b_i = \pm c i a_i$ where $c \in \mathbb{R}$. since it follows that
$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 = \left| \sum_{j=1}^{n} \pm a_j \, c i a_j \right|^2$$
$$= c^2 \left| \sum_{j=1}^{n} a_j \bar{a}_j \right|^2$$
$$= c^2 \left| \sum_{j=1}^{n} |a_j|^2 \right|^2$$
$$= c^2 \sum_{j=1}^{n} |a_j|^2.$$

Likewise,
$$\sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 = \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |c i b_j|^2 = c^2 \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

(d) Computing we have that
$$\sum_{j=1}^{n} |a_j + b_j|^2 = \sum_{j=1}^{n} |a_j|^2 + 2\,\mathrm{Re}(a_j \bar{b}_j) + |b_j|^2$$
$$\leq \sum_{j=1}^{n} |a_j|^2 + 2\left(\sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 \right)^{1/2} + \sum_{j=1}^{n} |b_j|^2$$
$$= \left[\left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \right]^2$$
$$\Longrightarrow \sum_{j=1}^{n} |a_j + b_j|^2 \leq \sum_{j=1}^{n} |a_j|^2 + \sum_{j=1}^{n} |b_j|^2.$$

2. Let $z = x+iy$. Prove that $|z| \leq |x| + |y|$.

~~proof~~

$$|z|^2 = x^2 + y^2$$
$$\leq x^2 + 2|x|\cdot|y| + y^2$$
$$= (|x|+|y|)^2$$
$$\Longrightarrow |z| \leq |x| + |y|.$$

$\blacksquare$