# MTH 317/617
# Homework #4

Due Date: October 6, 2023

## 1 Problems for Everyone

1. Show that each of the following functions is nowhere differentiable using the definition of the derivative, i.e. do not use the Cauchy Riemann equations.
   (a) $f(z) = \text{Re}(z)$
   (b) $f(z) = \text{Im}(z)$
2. Let $P(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. Prove that
$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_n}$$
3. Find real constants $a, b, c, d$ so that the given function is analytic
   (a) $f(z) = 3x - y + 5 + i(ax + by - 3)$.
   (b) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$.
4. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic then
$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2.$$
5. Verify that each given function $u$ is harmonic (in the region where it is defined) and then find the function $v$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic.
   (a) $u = e^x \sin(y)$
   (b) $u = xy - x + y$
   (c) $u = \sin(x)\cosh(y)$
   (d) $u = \text{Im}\left(\exp(z^2)\right)$

## 2 Graduate Problems

1. A set $A$ is said to be an ordered set provided it contains a subset $P$ with the following two properties
   - For any $x \in A$, either $x \in P$ or $-x \in P$ (but not both).
   - If $x, y \in P$ then both $xy \in P$ and $x + y \in P$.

   In $\mathbb{R}$ the set $P$ is the set of positive numbers. In $\mathbb{R}$ we say $x > y$, if and only if $x - y \in P$. Prove that $\mathbb{C}$ is not an ordered set.

Homework #4

#1.

Show that each of the following function is nowhere differentiable.

(a) $f(z) = Re(z)$

(b) $f(z) = Im(z)$

Solution:

(a) Let $z_0 = x_0 + iy_0 \in \mathbb{C}$, $\Delta z_n = 1/n$, and $\Delta w_n = i/n$. Therefore,

$$\frac{f(z_0+\Delta z_n) - f(z_0)}{\Delta z_n} = \frac{x_0 + 1/n - x_0}{1/n} = 1$$

$$\frac{f(z_0+\Delta w_n) - f(z_0)}{\Delta w_n} = \frac{x_0 - x_0}{i/n} = 0.$$

Therefore, $f(z) = Re(z)$ is not differentiable.

(b) Let $z_0 = x_0 + iy_0 \in \mathbb{C}$, $\Delta z_n = 1/n$, and $\Delta w_n = i/n$. Therefore,

$$\frac{f(z_0+\Delta z_n) - f(z_0)}{\Delta z_n} = \frac{\cancel{y_0} - \cancel{y_0}}{1/n} = 0$$

$$\frac{f(z_0+\Delta w_n) - f(z_0)}{\Delta w_n} = \frac{\cancel{y_0} + 1/n - \cancel{y_0}}{i/n} = -i.$$

Therefore, $f(z) = Im(z)$ is not differentiable.

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#2.

Let $P(z) = (z - z_1) \cdots (z - z_n)$. Prove that

$$\frac{P'(z)}{P(z)} = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_n}.$$

proof:

By the product rule we have that

$$p'(z) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} (z - z_j)$$

Therefore,

$$\frac{p'(z)}{p(z)} = \frac{\sum_{i=1}^{n} \prod_{j \neq i}^{n} (z - z_j)}{\prod_{k=1}^{n} (z - z_k)}$$

$$= \sum_{i=1}^{n} \frac{1}{(z - z_i)}$$

$$= \frac{1}{z - z_1} + \dots + \frac{1}{z - z_k}$$

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#3 Find real constants so that the given function is analytic.

(a) $f(z) = 3x - y + 5 + i(ax + by - 3)$

(b) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

Solution:

(a) Letting $u(x,y) = 3x - y + 5$ and $v(x,y) = ax + by - 3$. We have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 3 = b$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow 1 = a$$

(b) Letting $u(x,y) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$. We have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x + ay = dx + 2y \Rightarrow d = 2,\ a = 2.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2x + 2by = -2cx - 2y \Rightarrow c = -1,\ b = -1.$$

■

## #4

Prove that if $f(z) = u(x,y) + iv(x,y)$ is analytic then
$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2.$$

*proof:*

Differentiating, we have that
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad f'(z) = -\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y}.$$

Therefore,
$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

$\blacksquare$

## #5

Verify that each function is harmonic and then find the function $v$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic.

(a) $u = e^x \sin(y)$

(b) $u = xy - x + y$

(c) $u = \sin(x)\cosh(y)$

(d) $u = \operatorname{Im}(\exp(z^2))$.

Solution:

(a) Computing we have that
$$u_x = e^x \sin(y), \quad u_y = e^x \cos(y)$$
$$u_{xx} = e^x \sin(y) \qquad u_{yy} = -e^x \sin(y)$$

and thus $u_{xx} + u_{yy} = 0$. Now, by the Cauchy-Riemann equations we have that
$$e^x \sin(y) = v_y$$
$$\Rightarrow v = -e^x \cos(y) + \varphi(x)$$
$$\Rightarrow -e^x \cos(y) = -e^x \cos(y) + \varphi'(x).$$

Therefore,

$$v(x,y) = -e^x \cos(y)$$

works.

(b) Computing, we have that

$$u_x = y-1 \qquad u_y = x+1$$

$$u_{xx} = 0 \qquad u_{yy} = 0$$

Consequently, $u$ is harmonic. Furthermore,

$$y-1 = v_y$$

$$\Rightarrow \frac{y^2}{2} - y + \phi(x) = v$$

$$\Rightarrow \phi'(x) = -x-1.$$

$$\Rightarrow \phi(x) = -\frac{x^2}{2} - x$$

Therefore,

$$v = \frac{y^2}{2} - y - \frac{x^2}{2} - x$$

works.

(c) Computing, we have that

$$u_x = \cos(x)\cosh(y) \qquad u_y = \sin(x)\sinh(y)$$

$$u_{xx} = -\sin(x)\cosh(y) \qquad u_{yy} = \sin(x)\cosh(y)$$

and thus $u$ is harmonic. Furthermore,

$$\cos(x)\cosh(y) = v_y$$

$$\Rightarrow v = \cos(x)\sinh(y) + \varphi(x)$$

$$\Rightarrow -\sin(x)\sinh(y) = -\sin(x)\sinh(y) + \varphi'(x).$$

$$\Rightarrow \varphi(x) = 0.$$

Therefore,

$$v = \cos(x)\sinh(y)$$

works.

(d) We have that

$$v = \operatorname{Im}(e^{z^2})$$
$$= \operatorname{Re}(-i e^{z^2}).$$

Therefore, $v$ is harmonic since $-ie^{z^2}$ is analytic.

Furthermore,

$$v = \operatorname{Im}(-i e^{z^2})$$
$$= \operatorname{Im}(-i e^{(x+iy)^2})$$
$$= \operatorname{Im}(-i e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)))$$
$$= -e^{x^2-y^2}\cos(2xy).$$

∎

#1.

Prove that $\mathbb{C}$ is not an ordered set.

Solution:

Suppose for contradiction that $\mathbb{C}$ is ordered. Therefore, there exists $P \subset \mathbb{C}$ such that

(i) For any $z \in \mathbb{C}$, either $z \in P$ or $-z \in P$ but not both.

(ii) If $z, w \in P$ then $zw \in P$ and $z + w \in P$.

Therefore, $i \in P$ or $-i \in P$. If $\pm i \in P$ then $\pm i \cdot \pm i = -1 \in P$ and thus $-1 \cdot \pm i = \mp i \in P$ which is a contradiction.

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