MTH 317/617
Homework #6

Due Date: October 27, 2023

1 Problems for Everyone

1. For each of the following curves give an admissible parametrization that is consistent with the indicated direction.
   (a) The line segment from $z = 1 + i$ to $z = -2 - 3i$.
   (b) The circle $|z - 2i| = 4$ transversed once in the clockwise direction starting from $z = 4 + 2i$.
   (c) The arc of the circle $|z| = R$ lying in the second quadrant, from $z = Ri$ to $z = -R$.
   (d) The segment of the parabola $y = x^2$ from the point $(1, 1)$ to the point $(3, 9)$.
2. Using an admissible parametrization, verify from the arclength integral that
   (a) The length of the line segment from z_1 to z_2 is $|z_1 - z_2|$.
   (b) The length of the circle $|z - z_0| = r$ is $2\pi r$.
3. In class we showed for $n \in \mathbb{Z}$ and C a circle of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ that

$$\int_C (z - z_0)^n ds = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}.$$

   Utilize this fact to evaluate the following contour integral

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2 \right] dz,$$

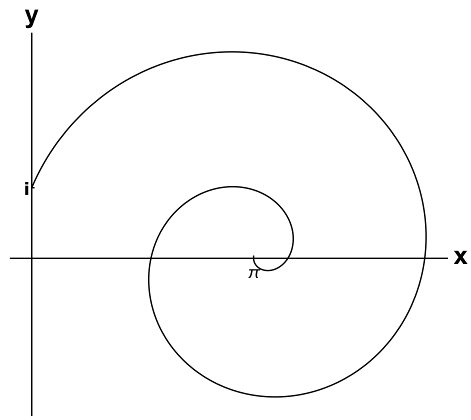
   where C is the circle $|z - i| = 4$ traversed once counterclockwise.
4. Let C be the perimeter of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$ traversed once in that order.
   (a) Show by explicitly parametrizing C and computing the contour integral that $\int_C z^2 dz = 0$.
   (b) Show by explicitly parametrizing C and computing the contour integral that $\int_C \overline{z}^2 dz \neq 0$. Why does this result not violate the independence of path theorem?

5. Let γ_1 be the semicircle from 1 to -1 that passes through i and γ_2 the semicircle from 1 to -1 that passes through $-i$.

   (a) Compute $\int_{\gamma_1} z dz$ and $\int_{\gamma_2} z dz$. Why are these results equal?

   (b) Compute $\int_{\gamma_1} \bar{z} dz$ and $\int_{\gamma_2} \bar{z} dz$. Why are these results not equal?

6. The contour Γ drawn below starts at $z = \pi$ and ends at $z = i$.



   Calculate the following integrals

   (a) $\displaystyle\int_\Gamma (3z^2 - 5z + i) dz$

   (b) $\displaystyle\int_\Gamma e^z dz$

   (c) $\displaystyle\int_\Gamma \sin^2(z)\cos(z) dz$

   (d) $\displaystyle\int_\Gamma e^z \cos(z) dz$

7. Compute the following integrals

   (a) $\int_\gamma z dz$, where γ is the semicircle from i to $-i$ which passes through -1.

   (b) $\int_\gamma e^z dz$, where γ is the line segment from 0 to z_0.

   (c) $\int_\gamma |z|^2 dz$, where γ is the line segment from 2 to $3 + i$.

   (d) $\int_\gamma 1/(4+z) dz$, where γ is the circle of radius 1 centered at -4, oriented counterclockwise.

   (e) $\int_\gamma \mathrm{Re}(z) dz$, where γ is the line segment from 1 to i.

   (f) $\int_\gamma (z^2 + 3z + 4) dz$ where γ is the circle $|z| = 2$ oriented counterclockwise.

2 Graduate Problems

1. Let $z = z_1(t)$ be an admissible parametrization of the smooth curve γ. If $\phi(s)$, $c \le s \le d$ is a differentiable function satisfying $\phi'(s)$ is continuous, and $\phi(c) = a, \phi(d) = b$, then the function $z_2(s) = z_1(\phi(s))$, $c \le s \le d$ is also an admissible parametrization of γ. Verify that

$$\int_a^b |z_1'(t)| dt = \int_c^d |z_2'(s)| ds.$$

Homework #6

#3

Evaluate the following contour integral

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2\right] dz$$

where C is the circle $|z-i|=4$ traversed once counterclockwise.

Solution:

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + (1-3(z-i)^2)\right] dz = 6\int_C \frac{1}{(z-i)^2} dz + 2\int_C \frac{1}{z-i} dz$$
$$+ \int_C dz - 3\int_C (z-i)^2 dz$$
$$= 0 + 4\pi i + 0 - 0$$
$$= 4\pi i.$$

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#4

Let C be the perimeter of the square with vertices at the points $z=0$, $z=1$, $z=1+i$, $z=i$ traversed once in that order.

(a) Show by explicitely parametrizing C and computing the contour integral that $\int_C z^2 dz = 0$.

(b) Show that $\int_C \bar{z}^2 dz \neq 0$.

Solution:

(a) We first parametrize the contour using 4 curves.

$$z_1(t) = t, \quad t\in[0,1] \quad \Rightarrow z_1'(t) = 1$$
$$z_2(t) = 1+it, \quad t\in[0,1] \Rightarrow z_2'(t) = i$$
$$z_3(t) = 1+i-t, \quad t\in[0,1] \Rightarrow z_3'(t) = -1$$
$$z_4(t) = i-it, \quad t\in[0,1] \Rightarrow z_4'(t) = -i$$

Therefore,

$$\begin{aligned}\int_C z^3\,dz &= \int_0^1 t^2\,dt + i\int_0^1 (1+it)^2\,dt - \int_0^1 (1+i-t)^2\,dt - i\int_0^1 (i-it)^2\,dt \\ &= \int_0^1 \cancel{t^2}\,dt + i\int_0^1 (1+2it-\cancel{t^2})\,dt - \int_0^1 \left((1+i)^2-2(1+i)t+\cancel{t^2}\right)dt + i\int_0^1 (1-2t+\cancel{t^2})\,dt \\ &= i\int_0^1 (\cancel{1}+2it)\,dt - \int_0^1 (\cancel{2i}-\cancel{2t}-2it)\,dt + i\int_0^1 (\cancel{1}-\cancel{2t})\,dt \\ &= 0.\end{aligned}$$

(b) We also have that

$$\begin{aligned}\int_C \bar{z}^2\,dz &= \int_0^1 t^2\,dt + i\int_0^1 (1-it)^2\,dt - \int_0^1 (1-i-t)^2\,dt - i\int_0^1 (-i+it)^2\,dt \\ &= \tfrac{1}{3} + \tfrac{i}{-i}(1-it)^3/3\Big|_0^1 + \tfrac{1}{3}(1-i-t)^3\Big|_0^1 - (-i+it)^3/3\Big|_0^1 \\ &= \tfrac{1}{3} - (1-i)^3/3 + \tfrac{1}{3} + \tfrac{1}{3}(-i)^3 - \tfrac{1}{3}(1-i)^3 - (-i+i)^3 + (-i)^3/3 \\ &= \tfrac{2}{3} + \tfrac{2i}{3} - \tfrac{2}{3}(1-i)^3 \\ &= 2+2i.\end{aligned}$$

This does not violate independence of path since $f(z)=\bar{z}^2$ is not analytic.

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#5

Let γ_1, γ_2 be semicircles from 1 to -1 passing through i and $-i$ respectively.

(a) Compute $\int_{\gamma_1} z\,dz$ and $\int_{\gamma_2} z\,dz$.

(b) Compute $\int_{\gamma_1} \bar{z}\,dz$ and $\int_{\gamma_2} \bar{z}\,dz$.

Solution:

(a) Since the antiderivative of z is $\frac{1}{2}z^2$ we have that

$$\int_{\gamma_1} z\,dz = \int_{\gamma_2} z\,dz = \frac{1}{2}z^2\Big|_1^{-1} = 0.$$

(b) We can parameterize γ_1, γ_2 by $z_1(t) = e^{it}, z_2 = e^{-it}$ $(t \in [0,\pi])$ respectively. Therefore,

$$\int_{\gamma_1} \bar{z}\,dz = \int_0^{\pi} i e^{-it} e^{it}\,dt = \pi i$$

$$\int_{\gamma_2} \bar{z}\,dz = \int_0^{\pi} -i e^{it} e^{-it}\,dt = -\pi i.$$

These values are not equal since $f(z) = \bar{z}$ is not analytic. ∎

#6

The contour Γ starts at $z = \pi$ and ends at $z = i$. Calculate the following integrals

(a) $\int_{\Gamma} (3z^2 - 5z + i)\,dz$

(b) $\int_{\Gamma} e^z\,dz$

(c) $\int_{\Gamma} \sin^2(z)\cos(z)\,dz$

(d) $\int_{\Gamma} e^z\cos(z)\,dz$

Solution:

(a) $\int_\Gamma (3z^2-5z+i)dz\ z^3-\frac{5}{2}z^2+iz\Big|_\pi^i = -i+5/2-1-\pi^3+\frac{5}{2}\pi^2-i\pi.$

(b) $\int_\Gamma e^z dz = e^z\Big|_\pi^i = e^i - e^\pi.$

(c) $\int_\Gamma \sin^2(z)\cos(z)dz = \frac{1}{3}\sin^3(z)\Big|_\pi^i = \frac{1}{3}\sin(i).$

(d) $\int_\Gamma e^z\cos(z)dz = \frac{1}{2}e^z(\cos(z)+\sin(z))\Big|_\pi^i = \frac{1}{2}e^i(\cos(i)+\sin(i)) - \frac{1}{2}e^\pi.$

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#7

Compute the following integrals

(b) $\int_\gamma e^z dz$, where γ is the line segment from 0 to z_0.

(c) $\int_\gamma |z|^2 dz$, where γ is the line segment from 2 to $3+i$

(e) $\int_\gamma Re(z)dz$, where γ is the line segment from 1 to i.

(f) $\int_\gamma (z^2+3z+4)dz$, where γ is the circle $|z|=2$ oriented counterclockwise.

Solution:

(b) $\int_\gamma e^z dz = e^z\Big|_0^{z_0} = e^{z_0}-1$

(c) We paramterize the curve by $z(t)=t(3+i)+(1-t)2$ for $t\in[0,1]$.
Therefore, $|z(t)|^2=(2+t)^2+t^2=2t^2+2t+4$ and $z'(t)=1+i$. Therefore,

$$\begin{aligned}\int_\gamma |z|^2dz &= \int_0^1(2t^2+4t+4)(1+i)dt\\ &= (2/3+2+4)(1+i)\\ &= \frac{20}{3}+\frac{20}{3}i\end{aligned}$$

(e) We parametrize the curve by $z(t)=(1+i)t$ for $t\in[0,1]$. Therefore,

$$\int_\gamma Re(z)dz = \int_0^1 t(1+i)dt = \frac{1}{2}(1+i).$$

(f) Since $f(z)=z^2+3z+4$ is analytic we have that

$$\int_\gamma (z^2+3z+4)dz = 0.$$

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Graduate Problem

#1

Let $z = z_1(t)$ be an admissable parametrization of the smooth curve γ. If $\phi(s)$, $c \leq s \leq d$ is a differentiable function satisfying $\phi'(s)$ is continuous, and $\phi(c) = a$, $\phi(d) = b$, then the function $z_2(s) = z_1(\phi(s))$, $c \leq s \leq d$ is also an admissiable parametrization of γ. Verify that

$$\int_a^b |z_1'(t)|\,dt = \int_c^d |z_2'(s)|\,ds.$$

Solution:

$$\int_c^d |z_2'(s)|\,ds = \int_c^d |z_1'(\phi(s))|\,ds$$
$$= \int_c^d \left| \frac{dz_1}{ds}\bigg|_{\phi(s)} \right| ds$$

Letting $t = \phi(s)$ we have that $dt = \phi'(s)\,ds$ and

$$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} = \phi'(s)\frac{d}{dt}.$$

Consequently,

$$\int_c^d |z_2'(s)|\,ds = \int_a^b \left| \frac{dz_1}{dt} \right|_t \frac{|\phi'(s)|}{\phi'(s)}\,dt$$
$$= \int_a^b |z_1'(t)|\,\mathrm{sgn}(\phi'(s))\,dt.$$

Assuming ϕ' is monotone we obtain the result. ∎