MTH 317/617
Homework #7

Due Date: November 10, 2023

1 Problems for Everyone

1. Determine the domain of analyticity for the following functions and explain why

$$\int_{|z|=2} f(z)dz = 0.$$

   (a) $f(z) = \dfrac{z}{z^2 + 25}$

   (b) $f(z) = \dfrac{\cos(z)}{z^2 - 6z + 10}$

   (c) $f(z) = \text{Log}(z + 3)$

   (d) $f(z) = \sec\left(\dfrac{z}{2}\right)$.

2. pg. 202, #13.

3. Evaluate

$$\int_\Gamma \frac{z}{(z+2)(z-1)} dz,$$

   where Γ is the circle $|z| = 4$ traversed once in the clockwise direction.

4. pg. 202, #16.

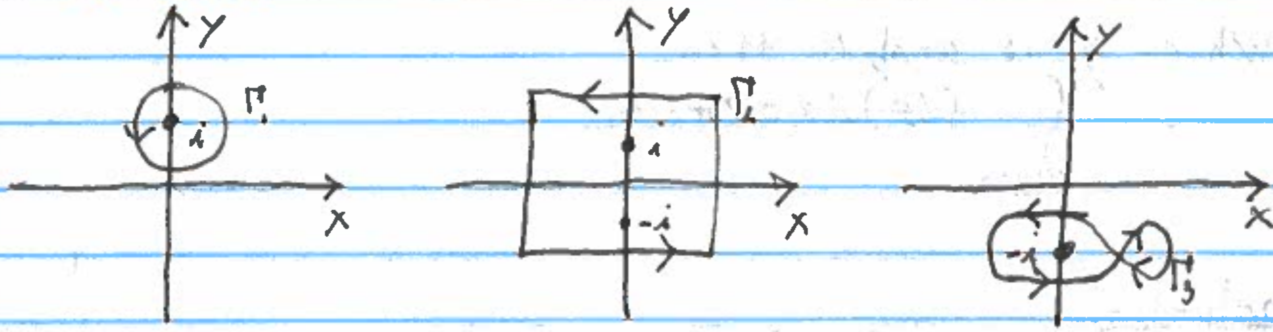
5. pg. 203, #17.

6. pg. 203, #18.

Homework #7

#2

Evaluate $\int_{\Gamma_i} 1/(z^2+1)\,dz$ along the three closed contours $\Gamma_1, \Gamma_2, \Gamma_3$ drawn below



Solutions:

Since $\int_{\Gamma_i} 1/(z^2+1)\,dz = \int_{\Gamma_i} 1/(z+i)(z-i)\,dz$ we have that:

- $\int_{\Gamma_1} 1/(z^2+1)\,dz = 2\pi i \cdot 1/(z+i)\big|_{z=i} = \pi.$
- $\int_{\Gamma_2} 1/(z^2+1)\,dz = 2\pi i \left(1/(z+i)\big|_{z=i} + 1/(z-i)\big|_{z=-i}\right) = 0$
- $\int_{\Gamma_3} 1/(z^2+1)\,dz = 2\pi i \left(1/(z-i)\right)\big|_{z=-i} = -\pi.$

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#3

Evaluate

$$\int_{\Gamma} z/(z+2)(z-1)\,dz,$$

where Γ is the circle transversed once in the clockwise direction.

Solution:

$$\int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz = 2\pi i\left(\frac{z}{z-1}\bigg|_{z=-2} + \frac{z}{z+2}\bigg|_{z=1}\right)$$

$$= 2\pi i\left(\frac{2}{3} + \frac{1}{3}\right) = 2\pi i.$$

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#4

Show that if f is of the form

$$f(z) = \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \dots + \frac{A_1}{z} + g(z),$$

Where g is analytic then

$$\int_{|z|=1} f(z)\,dz = 2\pi i A_1.$$

Solution:

$$\int_{|z|=1} f(z)\,dz = \int_{|z|=1} \left(\frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \dots + \frac{A_1}{z} + g(z) \right) dz$$

$$= \int_{|z|=1} \frac{A_k}{z^k}\,dz + \int_{|z|=1} \frac{A_{k-1}}{z^{k-1}}\,dz + \dots + \int_{|z|=1} \frac{A_1}{z}\,dz + \int_{|z|=1} g(z)\,dz$$

$$= A_k \int_{|z|=1} \frac{1}{z^k}\,dz + A_{k-1} \int_{|z|=1} \frac{1}{z^{k-1}} + \dots + A_1 \int_{|z|=1} \frac{1}{z}\,dz + 0$$

$$= A_k \cdot 0 + A_{k-1} \cdot 0 + \dots + 2\pi i A_1$$

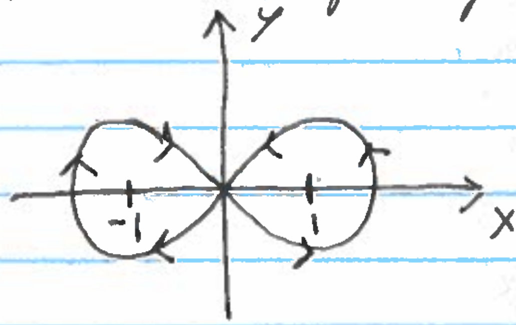
$$= 2\pi i A_1$$

#5

Evaluate

$$\int_{\Gamma} \frac{2z^2 - z + 1}{(z-1)^2(z+1)}\, dz$$

where Γ is the figure-eight contour drawn below:

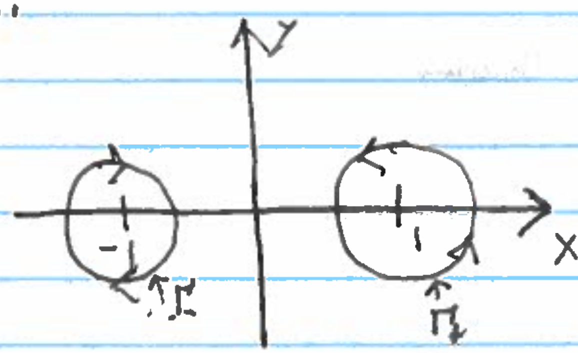


Solution:

The partial fraction decomposition is given by

$$\frac{2z^2 - z + 1}{(z-1)^2(z+1)} = \frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{z+1}.$$

We can deform the above contour into the equivalent form:



Therefore,

$$\int_{\Gamma} \frac{2z^2 - z + 1}{(z-1)^2(z+1)}\, dz = -\int_{\Gamma_1} \frac{1}{z+1}\, dz + \int \frac{1}{z-1}\, dz$$

$$= -2\pi i + 2\pi i$$

$$= 0.$$

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#6

Let $I = \int_{|z|=2} \frac{dz}{z^2(z-1)^3}$. Prove that $I = 0$.

proof:

For all $R > 2$ we can deform the contour so that

$$I = \int_{|z|=R} z^{-2}(z-1)^{-3}\,dz.$$

Therefore, for $R > 2$ we have that

$$|I| \leq \int_{|z|=R} \frac{|dz|}{|z|^2(|z-1|)^3}$$

$$\leq \int_{|z|=R} \frac{|dz|}{R^2(||z|-1|)^3}$$

$$= \int_{|z|=R} \frac{|dz|}{R^2(R-1)^3}$$

$$= \frac{2\pi R}{R^2(R-1)^3}$$

$$= \frac{2\pi}{R^1(R-1)^3}$$

Therefore, by the squeeze theorem

$$I = \lim_{R\to\infty} I = 0.$$

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