MTH 317/617: Homework #1

Due Date: September 08, 2023

1 Problems for Everyone

1. If $z = 1 + 2i$, $w = 2 - i$ and $\zeta = 4 + 3i$, write the following complex expressions in the form $a + bi$ where a and b are real numbers.
   (a) $z + 3w$,
   (b) $-2w + \bar{\zeta}$,
   (c) z^2,
   (d) $w^3 + w$,
   (e) $\text{Im}(\zeta^{-1})$,
   (f) w/z,
   (g) $\zeta^2 + 2\bar{\zeta} + 3$.

2. Solve the following equations for z. Express your answer in the form $z = a + bi$ where a and b are real numbers.
   (a) $z = 1 - zi$,
   (b) $\dfrac{z}{1+z} = 1 + 2i$,
   (c) $(\pi + i)z - 8z^2 = 0$,
   (d) $z^2 + i = 0$.

3. Describe the set of points $z \in \mathbb{C}$ that satisfy each of the following.
   (a) $|z - 1 + 1| = 3$,
   (b) $|z - 1| = |z + 1|$,
   (c) $|z| = \text{Re}(z) + 2$,
   (d) $2 < |z| < 6$,
   (e) $\text{Re}(z/(1 + i)) = 0$.

4. Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following:
   (a) $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$,
   (b) $\text{Re}(z) = (z + \overline{z})/2$ and $\text{Im}(z) = -i(z - \overline{z})/2$,
   (c) If k is an integer then $(\overline{z})^k = \overline{(z^k)}$,
   (d) $|z| = 1$ if and only if $1/z = \bar{z}$.
   (e) If $|z| = 1$ and $z \neq 1$, then $\text{Re}\left((1 - z)^{-1}\right) = 1/2$.

5. Find the argument of the following complex numbers and write each in the polar form $z = r(\cos(\theta) + i\sin(\theta))$.

   (a) $-1 + i$,

   (b) $1 + i\sqrt{3}$,

   (c) $-i$,

   (d) $(2 - i)^2$,

   (e) $|4 + 3i|$,

   (f) $\sqrt{2}/(1 + i)$,

   (g) $[(1 + i)/\sqrt{2}]^4$,

6. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.

   (a) $e^{-i\frac{\pi}{2}}$,

   (b) $\dfrac{e^{1+3\pi i}}{e^{-1+\frac{\pi i}{2}}}$,

   (c) $\dfrac{e^{3i} - e^{-3i}}{2i}$,

   (d) e^{e^i}.

2 Graduate Problems

1. In this exercise you will prove the Cauchy-Schwarz inequality for complex numbers.

   (a) Let B, C be nonnegative real numbers and suppose that

   $$0 \leq B - 2\mathrm{Re}(\bar{\lambda}A) + |\lambda|^2 C$$

   for all $\lambda \in \mathbb{C}$. Prove that $|A|^2 \leq BC$.

   (b) Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be complex numbers. Prove the following inequality:

   $$\left| \sum_{j=1}^{n} a_j \bar{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{i=1}^{n} |b_j|^2. \tag{1}$$

   Hint: For all $\lambda \in \mathbb{C}$, we have $0 \leq \sum_{j=1}^{n} |a_j - \lambda b_j|^2$.

   (c) When does equality hold in (1)?

   (d) Use (1) to prove that

   $$\left(\sum_{j=1}^{n} |a_j + b_j|^2 \right)^{1/2} \leq \left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2}$$

2. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Prove that $|z| \leq |x| + |y|$.