MTH 317/617
Homework #2

Due Date: September 15, 2023

1 Problems for Everyone

1. Sketch the following curves in the complex plane:
   (a) $z(t) = 3e^{it}, 0 \leq t \leq 2\pi$.
   (b) $z(t) = 2e^{-it}, 0 \leq t \leq \pi$.
   (c) $z(t) = e^{-(1+i)t}, 0 \leq t \leq 2\pi$.
   (d) $z(t) = e^{(1+i)t}, 0 \leq t \leq 2\pi$.

2. Find all the values of the following. Express your answers in the form $z = a+ib$ where $a, b \in \mathbb{R}$.
   (a) $(-16)^{1/4}$.
   (b) $i^{1/4}$.
   (c) $(1-\sqrt{3}i)^{1/3}$.
   (d) $\left(\dfrac{2i}{1+i}\right)^{1/6}$.
   (e) $i^{\sqrt{2}}$.
   (f) $(\sqrt{2})^{i}$.

3. Find all solutions $z \in \mathbb{C}$ to the following equation
$$(z+2)^5 + z^5 = 0.$$

4. Let $a_0, \ldots, a_n \in \mathbb{R}$ and assume $z^* \in \mathbb{C}$ is a root of the polynomial
$$p(z) = a_0 + a_1 z + \ldots a_n z^n.$$
   (a) Prove that for all $z \in \mathbb{C}$, $p(\overline{z}) = \overline{p(z)}$.
   (b) Prove that $\overline{z^*}$ is also a root of the polynomial.

5. Prove that for all $n \in \mathbb{N}$,
$$\left(\frac{1+i\tan(\theta)}{1-i\tan(\theta)}\right)^n = \frac{1+i\tan(n\theta)}{1-i\tan(n\theta)}.$$

6. Prove the following identity:
$$\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta).$$

7. A point z_0 in a set D is called an interior point of D if there some neighborhood centered at z_0 contained in D. Hence, D is open if and only if it contains all of its interior points.

   For each of the following sets in $\mathbb{C}$, (a) sketch the set in $\mathbb{C}$, (b) describe the interior and the boundary, (c) state whether the set is open or closed, (d) state whether the interior is connected, (e) state if the set is bounded or unbounded.

   (a) $A = \{z = x + iy \in \mathbb{C} : x \geq 2 \text{ and } y \leq 4\}$
   (b) $B = \{z \in \mathbb{C} : |z| < 1 \text{ or } |z - 3| \leq 1\}$
   (c) $C = \{z = x + iy \in \mathbb{C} : x^2 < y\}$
   (d) $D = \{z \in \mathbb{C} : \mathrm{Re}(z^2) = 4\}$
   (e) $E = \{z \in \mathbb{C} : z\bar{z} - 2 \geq 0\}$
   (f) $F = \{z \in \mathbb{C} : z^3 - 2z^2 + 5z - 4 = 0\}$
   (g) $G = \{z = x + iy \in \mathbb{C} : -\pi \leq y < \pi\}$

2 Graduate Problems

1. Prove the following statements.

   (a) The boundary of any set $D \subseteq \mathbb{C}$ is a closed set.
   (b) For $D \subseteq \mathbb{C}$, show that if $p \in D$, then p is either an interior point of D or a boundary point of D.
   (c) Show that a set $D \subseteq \mathbb{C}$ coincides with its boundary if and only if D is closed and D has no interior points.
   (d) Show that if $D \subseteq \mathbb{C}$ and $E \subseteq \mathbb{C}$ is a closed set containing D, then E must contain the boundary of D.
   (e) Show that if $D \subseteq \mathbb{C}$ and $S \subseteq \mathbb{C}$ is an open set that is subset of D, then S must be composed entirely of interior points of D.