MTH 317/617
Homework #3

Due Date: September 22, 2023

1 Problems for Everyone

1. Write each of the following functions in the form $w = u(x, y) + iv(x, y)$ and for each function find the domain of definition.

   (a) $f(z) = 3z^2 + 5z + i + 1$

   (b) $f(z) = 1/z$

   (c) $f(z) = \dfrac{z+i}{z^2+1}$

   (d) $f(z) = e^{3z}$

   (e) $f(z) = \dfrac{2z^2+3}{|z-3|}$

   (f) $f(z) = e^z + e^{-z}$

2. For the complex function $f(z) = e^z$:

   (a) Describe the domain of definition and the range.

   (b) Show that $f(-z) = -1/f(z)$.

   (c) Describe the image of the vertical line $\text{Re}(z) = 1$.

   (d) Describe the image of the horizontal line $\text{Im}(z) = \pi/4$.

   (e) Describe the image of the infinite strip $0 \leq \text{Im}(z) \leq \pi/4$.

3. Let $F(z) = z + i$, $G(z) = iz$, and $H(z) = 2z$. Sketch the image of the semi-circle:

$$S = \{z \in \mathbb{C} : |z| = 1, \text{Im}(z) > 0 \text{ and } \text{Re}(z) > 0\}$$

   under the following mappings:

   (a) $F(z)$

   (b) $G(z)$

   (c) $H(z)$

   (d) $G(F(z))$

   (e) $G(H(z))$

   (f) $H(F(z))$

   (g) $F(G(H(z)))$

4. Prove the sequence of complex numbers $z_n = x_n + iy_n$ converges to $z_0 = x_0 + iy_0$ if and only if x_n converges to x_0 and y_n converges to y_0.

5. Prove that the sequence of complex numbers $z_n \to z_0$ if and only if $\overline{z_n} \to \overline{z_0}$.

6. Prove that $z_n \to 0$ if and only if $|z_n| \to 0$.

7. Compute the following limits justifying all steps or prove that the limit does not exist.

   (a) $\lim_{z \to 0} \frac{\mathrm{Im}(z)}{z}$.

   (b) $\lim_{z \to 0} z e^{i\mathrm{Re}(z)}$.

   (c) $\lim_{z \to 0} e^{\frac{1}{z}}$.

   (d) $\lim_{z \to i} \frac{1}{z - i} - \frac{1}{z^2 + 1}$.