MTH 317/617
Homework #6

Due Date: October 27, 2023

1 Problems for Everyone

1. For each of the following curves give an admissible parametrization that is consistent with the indicated direction.

   (a) The line segment from $z = 1 + i$ to $z = -2 - 3i$.

   (b) The circle $|z-2i| = 4$ transversed once in the clockwise direction starting from $z = 4+2i$.

   (c) The arc of the circle $|z| = R$ lying in the second quadrant, from $z = Ri$ to $z = -R$.

   (d) The segment of the parabola $y = x^2$ from the point $(1, 1)$ to the point $(3, 9)$.

2. Using an admissible parametrization, verify from the arclength integral that

   (a) The length of the line segment from z_1 to z_2 is $|z_1 - z_2|$.

   (b) The length of the circle $|z - z_0| = r$ is $2\pi r$.

3. In class we showed for $n \in \mathbb{Z}$ and C a circle of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ that

$$\int_C (z - z_0)^n ds = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}.$$

   Utilize this fact to evaluate the following contour integral

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2 \right] dz,$$

   where C is the circle $|z - i| = 4$ traversed once counterclockwise.

4. Let C be the perimeter of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$ traversed once in that order.

   (a) Show by explicitly parametrizing C and computing the contour integral that $\int_C z^2 dz = 0$.

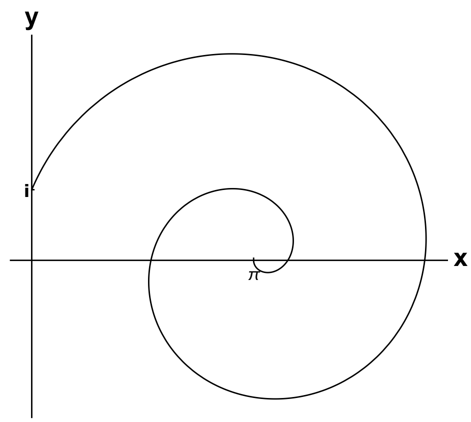
   (b) Show by explicitly parametrizing C and computing the contour integral that $\int_C \overline{z}^2 dz \neq 0$. Why does this result not violate the independence of path theorem?

5. Let γ_1 be the semicircle from 1 to -1 that passes through i and γ_2 the semicircle from 1 to -1 that passes through $-i$.

   (a) Compute $\int_{\gamma_1} zdz$ and $\int_{\gamma_2} zdz$. Why are these results equal?

   (b) Compute $\int_{\gamma_1} \bar{z}dz$ and $\int_{\gamma_2} \bar{z}dz$. Why are these results not equal?

6. The contour Γ drawn below starts at $z = \pi$ and ends at $z = i$.



   Calculate the following integrals

   (a) $\displaystyle\int_\Gamma (3z^2 - 5z + i)dz$

   (b) $\displaystyle\int_\Gamma e^z dz$

   (c) $\displaystyle\int_\Gamma \sin^2(z)\cos(z)dz$

   (d) $\displaystyle\int_\Gamma e^z \cos(z)dz$

7. Compute the following integrals

   (a) $\int_\gamma zdz$, where γ is the semicircle from i to $-i$ which passes through -1.

   (b) $\int_\gamma e^z dz$, where γ is the line segment from 0 to z_0.

   (c) $\int_\gamma |z|^2 dz$, where γ is the line segment from 2 to $3+i$.

   (d) $\int_\gamma 1/(4+z)dz$, where γ is the circle of radius 1 centered at -4, oriented counterclockwise.

   (e) $\int_\gamma \mathrm{Re}(z)dz$, where γ is the line segment from 1 to i.

   (f) $\int_\gamma (z^2 + 3z + 4)dz$ where γ is the circle $|z| = 2$ oriented counterclockwise.

2 Graduate Problems

1. Let $z = z_1(t)$ be an admissible parametrization of the smooth curve γ. If $\phi(s)$, $c \le s \le d$ is a differentiable function satisfying $\phi'(s)$ is continuous, and $\phi(c) = a, \phi(d) = b$, then the function $z_2(s) = z_1(\phi(s))$, $c \le s \le d$ is also an admissible parametrization of γ. Verify that

$$\int_a^b |z_1'(t)|dt = \int_c^d |z_2'(s)|ds.$$