Stochastic Calculus
Fall 2023
Exam 2
11/02/23

Name (Print): Key

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **You do not have to give explicit numbers for solutions to problems, i.e. you can leave the solution as a product and/or sum of numbers and you do not have to expand out terms with a factorial or binomial coefficients.**
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	10	
7	10	
Total:	100	

Do not write in the table to the right.

Throughout this exam you can use the following facts:

- If X is a Gaussian random variable with mean $\mu = 0$ and variance σ^2 then for $n \in \mathbb{N}$,

$$\mathbb{E}[X^n] = \begin{cases} \sigma^n (n-1)(n-3)\cdots 3 \cdots 1, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}.$$

- If X is a Gaussian random variable with mean $\mu \in \mathbb{R}$ and variance σ^2 then the probability density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.$$

- If X is a Gaussian random variable with mean $\mu \in \mathbb{R}$ and variance σ^2 then the moment generating function of X is given by

$$\phi(t) = e^{\mu t + \frac{\sigma^2}{2} t^2}.$$

1. (15 points) **(Short Answer)** Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

C **I** Let B_t be a standard Brownian motion. If $s < t$ then

$$\mathbb{E}[B_t^2 - B_s^2] = \mathbb{E}\left[(B_t - B_s)^2\right].$$

C **I** Let B_t be a standard Brownian motion and f a function such that $\mathbb{E}[f(X_t)] < \infty$ for all t. If $s < t$ then

$$\mathbb{E}\left[f(B_t) - f(B_s) | \sigma(B_s)\right] = \mathbb{E}\left[f(B_t) - f(B_s)\right],$$

where $\sigma(B_s)$ is the σ-algebra generated by B_s.

C **I** If X_t is a martingale with respect to the filtration $\mathcal{F}_t$ and g is a function such that $\mathbb{E}[g(X_t)^2] < \infty$, then $Y_t = g(X_t)$ is a martingale with respect to $\mathcal{F}_t$.

C **I** If X_t is a martingale with respect to the filtration $\mathcal{F}_t$ then $\mathbb{E}[X_t] = \mathbb{E}[X_0]$.

C **I** If X and Y are random variables on a probability space $(\Omega, \mathcal{F}, P)$ then

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}\left[\mathbb{E}[Y|X]\right].$$

2. (15 points) Suppose X_t is a Gaussian process with mean 0 and

$$\mathrm{Cov}(X_t, X_s) = ts.$$

For $\Delta t > 0$, let $\Delta X_t = X_{t+\Delta t} - X_t$ and $\Delta X_s = X_{s+\Delta t} - X_s$.

(a) (5 points) Compute $\mathrm{Var}[X_t]$.

$$\mathrm{Var}[X_t] = \mathrm{Cov}(X_t, X_t) = t^2.$$

(b) (5 points) Show that for all t,

$$\mathrm{Cov}(\Delta X_t, \Delta X_s) = \Delta t^2.$$

$$\begin{aligned}
\mathrm{Cov}(\Delta X_t, \Delta X_s) &= \mathrm{Cov}(X_{t+\Delta t} - X_t, X_{s+\Delta t} - X_s) \\
&= \mathrm{Cov}(X_{t+\Delta t}, X_{s+\Delta t}) - \mathrm{Cov}(X_{t+\Delta t}, X_s) \\
&\quad - \mathrm{Cov}(X_{s+\Delta t}, X_t) + \mathrm{Cov}(X_t, X_s) \\
&= (t+\Delta t)(s+\Delta t) - (t+\Delta t)s - (s+\Delta t)t + ts \\
&= (t+\Delta t)(\Delta t) - t(\Delta t) \\
&= \Delta t^2
\end{aligned}$$

(c) (5 points) **Short Answer:** Determine if ΔX_t and ΔX_s are independent.

Since $\mathrm{Cov}(\Delta X_t, \Delta X_s) = 0$ it follows that ΔX_t and ΔX_s are not independent.

3. (15 points) Let B_t be a standard Brownian motion. Compute and simplify the following:

(a) (5 points) $\mathbb{E}[B_t(B_s - B_r)]$ if $r \le s \le t$.

$$\mathbb{E}[B_t(B_s - B_r)] = \mathbb{E}[B_t B_s] - \mathbb{E}[B_t B_r]$$
$$= s - r.$$

(b) (5 points) $\mathbb{E}[B_s B_t^2]$ if $s \le t$.

$$\mathbb{E}[B_s B_t^2] = \mathbb{E}[B_s (B_t - B_s + B_s)^2]$$
$$= \mathbb{E}[B_s((B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2)]$$
$$= \mathbb{E}[B_s]\mathbb{E}[(B_t - B_s)^2] + 2\,\mathbb{E}[B_t - B_s]\mathbb{E}[B_s^2] + \mathbb{E}[B_s^3]$$
$$= 0(t-s) + 2 \cdot 0 \cdot s + 0$$
$$= 0$$

(c) (5 points) $\mathrm{Cov}(e^{-t}B_{e^{2t}}, e^{-s}B_{e^{2s}})$ if $s \le t$.

$$\mathrm{Cov}(e^{-t}B_{e^{2t}}, e^{-s}B_{e^{2s}}) = e^{-t}e^{-s}\,\mathrm{Cov}(B_{e^{2t}}, B_{e^{2s}})$$
$$= e^{-t}e^{-s}e^{2s}$$
$$= e^{-(t-s)}.$$

4. (15 points) Let B_t be a standard Brownian motion and for $t \in [0,1]$ define $Z_t = B_1 - B_{1-t}$.

(a) (5 points) **Short Answer:** What are the three properties that Z_t must satisfy in order for Z_t to have the same distribution as a Brownian motion.

1. $Z_0 = 0$
2. $\mathbb{E}[Z_t] = 0$
3. $Cov(Z_t, Z_s) = \min\{s,t\}$.

(b) (10 points) Show that Z_t has the same distribution as a Brownian motion on $[0,1]$.

1. $Z_0 = B_1 - B_1 = 0$
2. $\mathbb{E}[Z_t] = \mathbb{E}[B_1 - B_{1-t}] = \mathbb{E}[B_1] - \mathbb{E}[B_{1-t}] = 0$
3. Assuming $s \le t$ we have that

$$\begin{aligned} Cov(Z_t, Z_s) &= Cov(B_1 - B_{1-t}, B_1 - B_{1-s}) \\ &= Cov(B_1, B_1) - Cov(B_1, B_{1-s}) - Cov(B_1, B_{1-t}) + Cov(B_{1-t}, B_{1-s}) \\ &= 1 - (1-s) - (1-t) + (1-t) \\ &= s. \end{aligned}$$

By item 1~3, Z_t is a Brownian motion.

5. (20 points) Let B_t be a standard Brownian motion.

(a) (5 points) **Short Answer:** If M_t is stochastic process with the natural filtration $\mathcal{F}_t$ and $\mathbb{E}[|M_t|] < \infty$, what property must M_t satisfy in order to be a martingale?

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s.$$

(b) (10 points) Compute and simplify $\mathbb{E}[tB_t|\sigma(B_s)]$, where $\sigma(B_s)$ is the σ-algebra generated by B_s. Your answer should not have any expected values in it.

$$\begin{aligned}
\mathbb{E}[tB_t | \sigma(B_s)] &= t\,\mathbb{E}[B_t | \sigma(B_s)] \\
&= t\,\mathbb{E}[B_t - B_s + B_s | \sigma(B_s)] \\
&= t\,\mathbb{E}[B_t - B_s | \sigma(B_s)] + t\,\mathbb{E}[B_s | \sigma(B_s)] \\
&= t\,\mathbb{E}[B_t - B_s] + t B_s \\
&= t B_s
\end{aligned}$$

(c) (5 points) Determine if tB_t is a martingale with respect to $\sigma(B_s)$.

Since $\mathbb{E}[tB_t | \sigma(B_s)] \neq sB_s$, tB_t is not a martingale.

6. (10 points)

(a) (5 points) **Short Answer** For $X, Y \in L^2(\Omega, \mathcal{F}, P)$, state the Cauchy-Schwarz inequality.

$$|\langle X, Y\rangle| \leq \|X\| \cdot \|Y\|$$

(b) (10 points) For $X, Y \in L^2(\Omega, \mathcal{F}, P)$, prove the Cauchy-Schwarz inequality.

Let $f(t) = \|X - tY\|^2$. Therefore,

$$f(t) = \|X\|^2 - 2t\langle X, Y\rangle + t^2\|Y\|^2$$

$$\Rightarrow f'(t) = -2\langle X, Y\rangle + 2t\|Y\|.$$

Consequently, f is minimized when

$$t^* = \frac{\langle X, Y\rangle}{\|Y\|^2}.$$

Therefore $f(t^*) \geq 0$

$$\Rightarrow \|X\|^2 - \frac{2\langle X, Y\rangle^2}{\|Y\|^2} + \frac{2\langle X, Y\rangle^2}{\|Y\|^2} \geq 0$$

$$\Rightarrow \langle X, Y\rangle^2 \leq \|X\|^2 \cdot \|Y\|^2$$

$$\Rightarrow |\langle X, Y\rangle| \leq \|X\| \cdot \|Y\|.$$

7. (10 points)

(a) (5 points) **Short Answer** For $X, Y \in L^2(\Omega, \mathcal{F}, P)$, state the two properties that $\mathbb{E}[Y|X]$ must satisfy.

1. $\mathbb{E}[Y|X]$ is a function of X.
2. For all g such that $g(X) \in L^2$,
$$\mathbb{E}[g(X)Y] = \mathbb{E}[g(X)\mathbb{E}[Y|X]]$$

(b) (10 points) Prove that if X, Y are Gaussian random variables then

$$\mathbb{E}[Y|X] = \frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X.$$

1. Clearly $\frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X$ is a function of X.

2. Computing, we have that
$$\mathbb{E}\left[X \cdot \left(Y - \frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X\right)\right] = \mathbb{E}[XY] - \frac{\mathbb{E}[XY]\mathbb{E}[X^2]}{\mathbb{E}[X^2]}$$
$$= \mathbb{E}[XY] - \mathbb{E}[XY]$$
$$= 0.$$

Therefore, X and $Y - \frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X$ are independent. Consequently,
$$\mathbb{E}\left[g(X)\left(Y - \frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X\right)\right] = \mathbb{E}[g(X)]\,\mathbb{E}\left[Y - \frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X\right]$$
$$= \mathbb{E}[g(X)] \cdot 0$$
$$= 0.$$

$$\Rightarrow \mathbb{E}[g(X)Y] = \mathbb{E}\left[g(X)\frac{\mathbb{E}[YX]}{\mathbb{E}[X^2]}X\right].$$