MTH 383/683: Homework #2

Due Date: September 15, 2023

1 Problems for Everyone

1. **Gaussian Integration by Parts.** Let Z be a standard Gaussian random variable.

   (a) Show using integration by parts that for a differentiable function g,

   $$\mathbb{E}[Zg(Z)] = \mathbb{E}[g'(Z)].$$

   (b) Use this result to prove that for $j \in \mathbb{N}$,

   $$\mathbb{E}\left[Z^{2j}\right] = \frac{(2j)!}{2^j j!} = (2j-1)(2j-3)\cdots 5\cdot 3\cdot 1.$$

2. **MGF of Exponential Random Variables** Let X be a random variable with an exponential distribution with parameter $\lambda > 0$.

   (a) Show that the MGF of X is given by

   $$\phi(t) = \mathbb{E}\left[e^{tX}\right] = \frac{\lambda}{\lambda - t},\ t < \lambda.$$

   (b) Use $\phi(t)$ to compute the expectation and variance of X.

3. **Guassian Tail.** Consider a random variable X with finite MGF such that

   $$\phi(t) = \mathbb{E}\left[e^{\lambda X}\right] \le e^{t^2/2}$$

   for $\lambda > 0$. Using Chernoff's bound, prove that for $a > 0$,

   $$P(X > a) \le e^{-a^2/2}.$$

4. **Constructing a Random Variable from Another One.** Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1, 1]$. Find a function $f : [-1, 1] \mapsto \mathbb{R}^+$ such that $Y = f(X)$ has an exponential distribution with parameter $\lambda > 0$.

5. **Why $\sqrt{2\pi}$?** Use polar coordinates to prove that

   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dxdy = \pi.$$

   Conclude that this implies that

   $$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 1.$$

6. **Gaussian Random Variables**. Let Z be a standard Gaussian random variable.

(a) Show that for $\sigma > 0$ and $m \in \mathbb{R}$ the random variable $X = \sigma Z + m$ is also a Gaussian random variable with mean m and variance σ^2.

(b) Show that the moment generating function of a Gaussian random variable X with mean m and variance σ^2 is given by

$$\phi(t) = \mathbb{E}[e^t X] = e^{tm + t^2\sigma^2/2}.$$

Homework #2

#1

Let Z be a standard Gaussian random variable.

(a) Prove that for a differentiable g,

$$\mathbb{E}[Z\,g(Z)] = \mathbb{E}[g'(Z)]$$

(b) Prove for $j \in \mathbb{N}$,

$$\mathbb{E}[Z^{2j}] = \frac{(2j)!}{2^j j!} = (2j-1)(2j-3)\cdots 5\cdot 3\cdot 1.$$

Solution:

(a). $$\mathbb{E}[Z\,g(Z)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, z\, e^{-z^2/2} g(z)\,dz$$
$$= -\frac{1}{\sqrt{2\pi}} e^{-z^2/2} g(z)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} g'(z)\,dz$$
$$= \mathbb{E}[g'(Z)].$$

(b). If we assume for $i \in \mathbb{N}$ that $\mathbb{E}[Z^{2i}] = \frac{(2i)!}{2^i i!}$ it follows that

$$\mathbb{E}[Z^{2(i+1)}] = \mathbb{E}[Z\, Z^{2i+1}]$$
$$= (2i+1)\,\mathbb{E}[Z^{2i}]$$
$$= (2i+1)(2i-1)(2i-3)\cdots 5\cdot 3\cdot 1.$$

Moreover, since $\mathbb{E}[Z^2] = 1$ it follows from the principle of mathematical induction that for $j \in \mathbb{N}$, $\mathbb{E}[Z^{2j}] = (2j)!/2^j j!$ ∎

#2

Let X be a random variable with an exponential distribution with parameter $\lambda > 0$.

(a) Show that the MGF of X is given by

$$\phi(t) = \lambda/(\lambda - t), \quad t < \lambda$$

(b) Use $\phi(t)$ to compute the expectation and variance of X

Solution:

(a) $\phi(t) = \mathbb{E}[e^{tX}]$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \frac{\lambda}{t-\lambda} e^{tx-\lambda x} \Big|_0^\infty$$

$$= \frac{\lambda}{\lambda - t},$$

if $t \leq \lambda$.

(b) Computing, we have that

$$\phi'(t) = \frac{\lambda}{(\lambda-t)^2}, \quad \phi''(t) = \frac{2\lambda}{(\lambda-t)^3}.$$

Consequently,

$$\mathbb{E}[X] = 1/\lambda$$

$$\mathbb{E}[X^2] = 2/\lambda^2$$

and thus $\mathrm{Var}[X] = 2/\lambda^2$.

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#3.

Consider a random variable X with finite MGF such that

$$\phi(t) = \mathbb{E}[e^{tX}] \leq e^{t^2/2}$$

for $t>0$. Using Chernoff's bound, prove that for $a>0$,

$$P(X>a) \leq e^{-a^2/2}.$$

proof

$$P(X>a) \leq e^{-ta}\mathbb{E}[e^{tX}]$$

$$\leq e^{-ta} e^{t^2/2}$$

for all $t>0$. Setting $t=a$ we obtain

$$P(X>a) \leq e^{-a^2/2}.$$

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#4.

Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1,1]$. Find a function $f: [-1,1] \to \mathbb{R}^+$ such that $Y = f(X)$ has an exponential distribution with parameter $\lambda > 0$.

Solution:

Computing we have that

$$P(f(a) < Y < f(b)) = \int_{f(a)}^{f(b)} \lambda e^{-\lambda x}\, dx = e^{-\lambda f(a)} - e^{-\lambda f(b)}.$$

We also have that

$$P(f(a) < Y < f(b)) = P(a < X < b) = (b-a)/2.$$

Therefore, there exists a constant c such that

$$e^{-\lambda f(x)} + c = -x/2$$

$$\Rightarrow e^{-\lambda f(x)} = -x/2 - c$$

$$\Rightarrow f(x) = -\frac{1}{\lambda} \ln(-x/2 - c).$$

Assuming $f(-1) = 0$ and $f(1) = \infty$ we obtain $c = 1/2$ and thus

$$f(x) = -\frac{1}{\lambda} \ln\left(\tfrac{1}{2}(x-1)\right).$$

$\blacksquare$

#5.

Use polar coordinates to prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \pi.$$

Conclude that this implies that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1.$$

Solution:

Computing, we have that
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$
$$= 2\pi \cdot \tfrac{1}{2} e^{-r^2}\Big|_0^{\infty}$$
$$= \pi.$$

Therefore, since
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$$
$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2,$$
it follows that $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$. Finally,
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du = 1.$$

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#6

Let Z be a standard Gaussian random variable.

(a) Show that for $\sigma > 0$ and $m \in \mathbb{R}$ that the random variable $X = \sigma Z + m$ is also a Gaussian random variable with mean m and variance σ^2.

(b) Show that the moment generating function of a Gaussian random variable X with mean m and variance σ^2 is given by
$$\phi(t) = \mathbb{E}[e^{tX}] = e^{tm + t^2\sigma^2/2}.$$

Solution:

(a) Computing we have that
$$P(X < x) = P(\sigma Z + m < x)$$
$$= P(Z < x-m/\sigma)$$
$$= \int_{-\infty}^{x-m/\sigma} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds.$$

Therefore, the density is given by
$$\frac{dF_X(x)}{dx} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)$$

which is the density of a Gaussian random variable with mean m and variance σ^2.

(b) Computing we have that

$$\begin{aligned}
\mathbb{E}[e^{tX}] &= \mathbb{E}[e^{t\sigma Z + tm}] \\
&= e^{tm}\,\mathbb{E}[e^{t\sigma Z}] \\
&= e^{tm}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2 + t\sigma z}\,dz \\
&= e^{tm}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2 + t\sigma z - t^2\sigma^2/2 + t^2\sigma^2/2}\,dz \\
&= e^{tm}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t^2\sigma^2/2}\left(e^{-(z-t\sigma)^2/2}\right)dz \\
&= e^{tm + t^2\sigma^2/2},
\end{aligned}$$

where we have used the fact that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(z-t\sigma)^2/2}\,dz = 1.$$

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