MTH 383/683: Homework #4

Due Date: Octoberr 06, 2023

1 Problems for Everyone

1. **Fractional Brownian Motion** Fractional Brownian motion B_t^H, with index $H \in (0,1)$, is a Gaussian process with mean 0 and covariance

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $\Delta t > 0$, let $\Delta B_t^H = B_{t+\Delta t}^H - B_t^H$.

   (a) For $t > 0$, calculate $\mathrm{Var}[B_t^H]$ and write down the probability density of B_t^H.

   (b) Compute $\mathbb{E}[\Delta B_t^H]$ and $\mathrm{Var}[\Delta B_t^H]$.

   (c) Compute $\mathrm{Cov}(\Delta B_t^H, \Delta B_s^H)$ if $s \leq t$.

   (d) For fixed s and t, determine if the random variables ΔB_t^H and ΔB_s^H are independent.

2. **Brownian Bridge** Let B_t be a standard Brownian motion. The Brownian bridge is the stochastic process Z_t defined by

$$Z_t = B_t - tB_1$$

for $t \in [0,1]$. For $\Delta t > 0$, let $\Delta Z_t = Z_{t+\Delta t} - Z_t$.

   (a) Compute the exact values of Z_0 and Z_1.

   (b) Compute $\mathrm{Cov}(Z_t, Z_s)$ assuming $t > s$.

   (c) Compute $\mathrm{Var}(Z_t)$ and $\mathbb{E}[Z_t]$.

   (d) Compute $\mathbb{E}[\Delta Z_t]$ and $\mathrm{Var}[\Delta Z_t]$.

   (e) Compute $\mathrm{Cov}(\Delta Z_t, \Delta Z_s)$ if $s \leq t$.

   (f) For fixed s and t, determine if the random variables ΔZ_t and ΔZ_s are independent.

3. **Shifted Brownian Bridge** Let B_t be a standard Brownian motion. For $t, s \in [0,1]$, construct a Gaussian process S_t that satisfies the following properties:

   - $\mathrm{Cov}(S_t, S_s) = s(1-t)$ if $s \leq t$,
   - $S_0 = x_0$,
   - $S_1 = x_1$,

   where $x_0, x_1 \in \mathbb{R}$ are arbitrary.

Homework #4

#1 Fractional Brownian Motion:

Fractional Brownian motion B_t^H with $H \in (0,1)$ is a Gaussian process with mean 0 and covariance

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

For $\Delta t > 0$, let $\Delta B_t^H = B_{t+\Delta t}^H - B_t^H$.

(a) For $t > 0$ calculate $\mathrm{Var}[B_t^H]$ and the probability density for B_t^H.

(b) Compute $\mathbb{E}[\Delta B_t^H]$ and $\mathrm{Var}[\Delta B_t^H]$.

(c) Compute $\mathrm{Cov}(\Delta B_t^H, \Delta B_s^H)$ if $s \le t$.

(d) For fixed s and t, determine if $\Delta B_t^H, \Delta B_s^H$ are independent

Solution:

(a)
$$\begin{aligned}\mathrm{Var}[B_t^H] &= \mathrm{Cov}(B_t^H, B_t^H) \\ &= \frac{1}{2}\left(t^{2H} + t^{2H}\right) \\ &= t^{2H}.\end{aligned}$$

The density is therefore,

$$f(x) = \frac{1}{\sqrt{2\pi t^{2H}}} \exp\left(-\frac{x^2}{2t^{2H}}\right).$$

(b)
$$\begin{aligned}\mathbb{E}[\Delta B_t^H] &= \mathbb{E}[B_{t+\Delta t}^H - B_t^H] \\ &= \mathbb{E}[B_{t+\Delta t}^H] - \mathbb{E}[B_t^H] \\ &= 0.\end{aligned}$$

$$\begin{aligned}\mathrm{Var}[\Delta B_t^H] &= \mathrm{Cov}(\Delta B_t^H, \Delta B_t^H) \\ &= \mathrm{Cov}(B_{t+\Delta t}^H - B_t^H, B_{t+\Delta t}^H - B_t^H) \\ &= \mathrm{Var}[B_{t+\Delta t}^H] - 2\,\mathrm{Cov}(B_{t+\Delta t}^H, B_t^H) + \mathrm{Var}[B_t^H] \\ &= (t+\Delta t)^{2H} - \left((t+\Delta t)^{2H} + t^{2H} - \Delta t^{2H}\right) + t^{2H} \\ &= \Delta t^{2H}.\end{aligned}$$

(c) $\mathrm{Cov}(\Delta B_t^H, \Delta B_s^H) = \mathrm{Cov}(B_{t+\Delta t}^H - B_t^H, B_{s+\Delta t}^H - B_s^H)$

$$= \mathrm{Cov}(B_{t+\Delta t}^H, B_{s+\Delta t}^H) - \mathrm{Cov}(B_t^H, B_{s+\Delta t}^H) - \mathrm{Cov}(B_s^H, B_{t+\Delta t}^H) + \mathrm{Cov}(B_t^H, B_s^H)$$

$$= \frac{1}{2}\left(\cancel{(t+\Delta t)^{2H}} + \cancel{(s+\Delta t)^{2H}} - |t-s|^{2H}\right)$$

$$- \frac{1}{2}\left(\cancel{t^{2H}} + \cancel{(s+\Delta t)^{2H}} - |t-s-\Delta t|^{2H}\right)$$

$$- \frac{1}{2}\left(\cancel{s^{2H}} + \cancel{(t+\Delta t)^{2H}} - |s-t-\Delta t|^{2H}\right)$$

$$+ \frac{1}{2}\left(\cancel{t^{2H}} + \cancel{s^{2H}} - |t-s|^{2H}\right)$$

$$= \frac{1}{2}|t-s-\Delta t|^{2H} + \frac{1}{2}|t-s+\Delta t|^{2H} - |t-s|^{2H}.$$

(d) ΔB_t^H, ΔB_s^H are dependent unless $H = 1/2$. ■

#2 Brownian Bridge:

Let B_t be standard Brownian motion. The Brownian bridge is the stochastic process defined by

$$Z_t = B_t - tB_1$$

for $t \in [0,1]$. For $\Delta t > 0$, let $\Delta Z_t = Z_{t+\Delta t} - Z_t$.

(a) Compute the exact values of Z_0 and Z_1.

(b) Compute $\mathrm{Cov}(Z_t, Z_s)$ assuming $t > s$.

(c) Compute $\mathrm{Var}(Z_t)$ and $\mathbb{E}[Z_t]$.

(d) Compute $\mathbb{E}[\Delta Z_t]$ and $\mathrm{Var}[\Delta Z_t]$.

(e) Compute $\mathrm{Cov}(\Delta Z_t, \Delta Z_s)$ if $s \leq t$.

(f) For fixed s and t, determine if ΔZ_t and ΔZ_s are independent.

Solution:

(a) $Z_0 = B_0 - 0 \cdot B_1 = 0$

$Z_1 = B_1 - 1 \cdot B_1 = 0$

(b) $Cov(Z_t, Z_s) = Cov(B_t - t B_1, B_s - s \cdot B_1)$

$= Cov(B_t, B_s) - t\,Cov(B_1, B_s) - s\,Cov(B_1, B_t) + st\,Var(B_1)$

$= s - ts - st + st$

$= s(1-t).$

(c) $Var(Z_t) = Cov(Z_t, Z_t) = t(1-t).$

$\mathbb{E}[Z_t] = \mathbb{E}[B_t - t B_1] = \mathbb{E}[B_t] - t\,\mathbb{E}[B_1] = 0.$

(d) $\mathbb{E}[\Delta Z_t] = \mathbb{E}[Z_{t+\Delta t} - Z_t]$

$= \mathbb{E}[Z_{t+\Delta t}] - \mathbb{E}[Z_t]$

$= 0$

$Var(\Delta Z_t) = Cov(Z_{t+\Delta t} - Z_t, Z_{t+\Delta t} - Z_t)$

$= Var(Z_{t+\Delta t}) - 2\,Cov(Z_t, Z_{t+\Delta t}) + Var(Z_t)$

$= (t + \Delta t)(1 - t - \Delta t) - 2t(1 - t - \Delta t) + t(1 - \Delta t)$

$= (1 - t - \Delta t)(\Delta t - t) + t(1 - \Delta t)$

$= \Delta t - \cancel{t\Delta t} - \Delta t^2 \cancel{- t} + t^2 \cancel{+ t\Delta t} \cancel{+ t} - t\Delta t$

$= \Delta t - \Delta t^2 + t^2 - t\Delta t$

$= \Delta t(1 - \Delta t) + t(1 - \Delta t)$

$= (1 - \Delta t)(t + \Delta t).$

(e) $Cov(\Delta Z_t, \Delta Z_s) = Cov(Z_{t+\Delta t} - Z_t, Z_{s+\Delta t} - Z_s)$

$= Cov(Z_{t+\Delta t}, Z_{s+\Delta t}) - Cov(Z_{t+\Delta t}, Z_s) - Cov(Z_t, Z_{s+\Delta t})$

$+ Cov(Z_t, Z_s)$

$= (s + \Delta t)(1 - t - \Delta t) - s(1 - t - \Delta t)$

$- (s + \Delta t)(1 - t) + s(1 - t)$

$= \Delta t(1 - t - \Delta t) - \Delta t(1 - t)$

$= -\Delta t^2.$

#3 Shifted Brownian Bridge

Let B_t be standard Brownian motion. For $t, s \in [0,1]$ construct a Gaussian process S_t that satisfies

(a) $\mathrm{Cov}(S_t, S_s) = s(1-t), \ s \le t$

(b) $S_0 = x_0$

(c) $S_1 = x_1$

Where $x_0, x_1 \in \mathbb{R}$ are arbitrary.

Solution:

Consider the process

$$S_t = B_t + x_0 - t(B_1 + x_0) + t x_1.$$

Therefore,

$$\begin{aligned}
\mathrm{Cov}(S_t, S_s) &= \mathrm{Cov}(B_t + x_0 - t(B_1 + x_0) + t x_1, \ B_t + x_0 - t(B_1 + x_0) + t x_1) \\
&= \mathrm{Cov}(B_t - t B_1, \ B_t - t B_1) \\
&= s(1-t)
\end{aligned}$$

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