MTH 383/683: Homework #6

Due Date: October 27, 2023

1 Problems for Everyone

1. **Conditional Expectation of Continuous Random Variables** Let X, Y be two random variables with joint density $f(x,y)$ on $\mathbb{R}^2$ and assume $f(x,y) > 0$ for all $x, y \in \mathbb{R}$. Show that the conditional expectation $\mathbb{E}[Y|X]$ equals $h(X)$ where h is the function

$$h(x) = \frac{\int_{-\infty}^{\infty} y f(x,y) dy}{\int_{-\infty}^{\infty} f(x,y)} dy.$$

   Hint: To prove this you need to show both properties of conditional expectation.

2. **Exercises on σ-fields** The Borel sets of $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra on $\mathbb{R}$ containing intervals of the form $(a, b]$. That is, $\mathcal{B}(\mathbb{R})$ contains all possible unions and intersections of intervals of the form $(a, b]$.

   (a) Show that all singletons $\{b\}$ are in $\mathcal{B}(\mathbb{R})$ by writing $\{b\}$ as the infinite intersection of intervals of the form $(b - 1/n, b + 1/n]$.

   (b) Prove that all open intervals (a, b) and closed intervals $[a, b]$ are in $\mathcal{B}(\mathbb{R})$.

3. **Another Look at Conditional Expectation for Gaussians** Let (X, Y) be a Gaussian vector with mean 0 and covariance matrix

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

   where $\rho \in (-1, 1)$.

   (a) Use Equation (4.5) in the text to show that $\mathbb{E}[Y|X] = \rho X$.

   (b) Write down the joint PDF $f(x,y)$ of (X, Y).

   (c) Show that

$$\int_{-\infty}^{\infty} y f(x,y) dy = \rho x \text{ and } \int_{-\infty}^{\infty} f(x,y) dy = 1.$$

   (d) Use problem #1 on this homework to show that $\mathbb{E}[Y|X] = \rho X$.

4. **Gaussian Conditioning** Consider the Gaussian vector (X_1, X_2, X_3) with mean 0 and covariance matrix

$$C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Prove that X_3 is independent of X_2 and X_1.

   (b) Compute $\mathbb{E}[X_2|X_1]$.

   (c) Write X_2 as a linear combination of X_1 and a random variable independent of X_1.

   (d) Compute $\mathbb{E}[e^{aX_2}|X_1]$ for any $a \in \mathbb{R}$.

5. Let B_t be a standard Brownian motion. Verify that $M_t = B_t^2 - t$ is a martingale for the Brownian filtration.

Homework #6

#1

Let X, Y be two random variables with joint density $f(x,y)$ on $\mathbb{R}^2$ and assume $f(x,y)>0$ for all $x,y\in\mathbb{R}$. Show that

$$\mathbb{E}[Y|X]=h(X)$$

where

$$h(x)=\frac{\int_{-\infty}^{\infty} y f(x,y)\,dy}{\int_{-\infty}^{\infty} f(x,y)\,dy}$$

Solution:

Let $g(X)$ be any function. Therefore,

$$\mathbb{E}[g(X)\,\mathbb{E}[Y|X]]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\left(\frac{\int_{-\infty}^{\infty} z f(x,z)\,dz}{\int_{-\infty}^{\infty} f(x,w)\,dw}\right) f(x,y)\,dy\,dx$$

$$=\int_{-\infty}^{\infty} g(x)\frac{\int_{-\infty}^{\infty} z f(x,z)\,dz}{\cancel{\int_{-\infty}^{\infty} f(x,w)\,dw}}\cancel{\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)}\,dx$$

$$=\int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} z f(x,z)\,dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,z f(x,z)\,dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,y f(x,y)\,dy$$

$$=\mathbb{E}[g(X)Y].$$

Thus,

$$\mathbb{E}[Y|X]=\mathbb{E}[g(X)Y].$$

$\blacksquare$

#2.

The Borel sets of $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra on $\mathbb{R}$ containing intervals of the form $(a,b]$.

(a) Show that singletons $\{b\}$ are in $\mathcal{B}(\mathbb{R})$.

(b) Prove that all open intervals (a,b) and closed intervals $[a,b]$ are in $\mathcal{B}(\mathbb{R})$.

Solution:

(a) Since $\{b\} = \bigcap_{n=1}^{\infty} (b-1/n, b+1/n]$ it follows that $\{b\} \in \mathcal{B}(\mathbb{R})$.

(b) Since $[a,b] = (a,b] \cup \{a\}$ and $(a,b) = (a,b] \cap \{b\}^c$ it follows that $(a,b), [a,b] \in \mathcal{B}(\mathbb{R})$.

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#3.

Let (X,Y) be a Gaussian vector with mean 0 and covariance matrix

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

where $\rho \in (-1,1)$.

(a) Show that $\mathbb{E}[Y|X] = \rho X$.

(b) Write down the joint P.D.F. of (X,Y).

(d) Use problem #1 to show that $\mathbb{E}[Y|X] = \rho X$.

Solution:

(a) $\mathbb{E}[Y|X] = \dfrac{\mathbb{E}[YX]}{\mathbb{E}[X^2]} X = \dfrac{\rho}{1} X = \rho X.$

(b). $f(x,y) = \dfrac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}$

(c) Computing, we have that

$$\int_{-\infty}^{\infty} y f(x,y)\,dy = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} y e^{-(x^2-2\rho xy+y^2)/1-\rho^2}\,dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} y e^{-x^2/1-\rho^2} e^{-(y^2-2\rho xy+\rho^2x^2)/1-\rho^2} e^{\rho^2x^2/1-\rho^2}\,dy$$

$$= \frac{e^{-(1-\rho^2)x^2}}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-(y-\rho x)^2/1-\rho^2}\,dy$$

$$= \frac{e^{-(1-\rho^2)x^2}}{\sqrt{2\pi}}\,\rho x.$$

$$\int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-(1-\rho^2)x^2} e^{-(y-\rho x)^2/1-\rho^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-(1-\rho^2)x^2}$$

(d) From problem #1 we have that

$$\mathbb{E}[Y|X] = h(X),$$

where

$$h(x) = \frac{\int_{-\infty}^{\infty} y f(x,y)\,dy}{\int_{-\infty}^{\infty} f(x,y)\,dy}$$

$$= \rho x.$$

Therefore,

$$\mathbb{E}[Y|X] = \rho X.$$

$\blacksquare$

#4.

Consider the Gaussian vector (X_1, X_2, X_3) with mean 0 and covariance matrix

$$C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Prove that X_3 is independent of X_2 and X_1

(b) Compute $\mathbb{E}[X_2 | X_1]$

(c) Write X_2 as a linear combination of X_1 and a random variable independent of X_1.

(d) Compute $\mathbb{E}[e^{aX_2} | X_1]$ for any $a \in \mathbb{R}$.

Solution:

(a) Since $\mathrm{Cov}(X_3, X_2) = \mathrm{Cov}(X_3, X_1) = 0$ it follows that X_3 is independent of X_1 and X_2.

(b). $$\mathbb{E}[X_2 | X_1] = \frac{\mathbb{E}[X_2 X_1]}{\mathbb{E}[X_1^2]} X_1 = \frac{2}{2} X_1 = X_1$$

(c) Using the orthogonal decomposition we have that

$$X_2 = (X_2 - \mathbb{E}[X_2 | X_1]) + \mathbb{E}[X_2 | X_1]$$
$$= (X_2 - X_1) + X_1$$

(d). $$\mathbb{E}[e^{aX_2} | X_1] = \mathbb{E}[e^{a(X_2 - X_1)} e^{aX_1} | X_1]$$
$$= e^{aX_1} \mathbb{E}[e^{a(X_2 - X_1)} | X_1]$$
$$= e^{aX_1} \mathbb{E}[e^{a(X_2 - X_1)}]$$

Now, $X_2 - X_1$ is Gaussian with mean 0 and variance

$$\sigma^2 = \mathbb{E}[(X_2 - X_1)^2]$$
$$= \mathbb{E}[X_2^2] - 2\mathbb{E}[X_2 X_1] + \mathbb{E}[X_1^2]$$
$$= 4 - 4 + 2$$
$$= 2.$$

Therefore, $\mathbb{E}[e^{aX_2} | X_1] = e^{aX_1} e^{a^2\sigma^2/2} = e^{aX_1 + a^2}$.

#5

Let B_t be a standard Brownian motion. Verify that $M_t = B_t^2 - t$ is a martingale for the Brownian filtration.

Solution:

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\begin{aligned}
\mathbb{E}[M_t \mid \sigma(B_s)] &= \mathbb{E}[B_t^2 - t \mid \sigma(B_s)] \\
&= \mathbb{E}[(B_t - B_s + B_s)^2 \mid \sigma(B_s)] - t \\
&= \mathbb{E}[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 \mid \sigma(B_s)] - t \\
&= \mathbb{E}[(B_t - B_s)^2 \mid \sigma(B_s)] + 2\mathbb{E}[(B_t - B_s)B_s \mid \sigma(D_s)] \\
&\quad + \mathbb{E}[B_s^2 \mid \sigma(B_s)] - t \\
&= \mathbb{E}[(B_t - B_s)^2] + B_s\,\mathbb{E}[B_t - B_s] + B_s^2 - t \\
&= t - s + 0 + B_s^2 - t \\
&= B_s^2 - s.
\end{aligned}
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