MTH 383/683: Homework #7

Due Date: November 10, 2023

1 Problems for Everyone

1. **Another Brownian Martingale:** Let B_t be standard Brownian motion. Consider for $a, b > 0$ the stopping time
$$\tau = \min_{t \geq 0}\{t : B_t \geq a \text{ or } B_t \leq -b\}.$$
   (a) Show that $M_t = tB_t - \frac{1}{3}B_t^3$ is a martingale for the Brownian filtration.
   (b) Use (a) to show that
$$\mathbb{E}[\tau B_\tau] = \frac{ab}{3}(a - b).$$

2. **Martingale Transform:** Let B_t be a standard Brownian motion on the interval $[0, 1]$ and let I_t be the stochastic process defined by
$$I_t = \begin{cases} 10B_t & t \in [0, 1/3] \\ 10B_{1/3} + 5(B_t - B_{1/3}) & t \in [1/3, 2/3] \\ 10B_{1/3} + 5(B_{2/3} - B_{1/3}) + 2(B_t - B_{2/3}) & t \in [2/3, 1] \end{cases}.$$
   (a) Show that I_t is martingale with respect to $\sigma(B_t)$.
   (b) Compute $\mathbb{E}[I_t^2]$.

3. **Ito Integral of Simple Process:** Let B_t be a standard Brownian motion and let I_t be the stochastic process defined by
$$I_t = \begin{cases} 0 & \text{if } s \in [0, 1/3] \\ B_{1/3}(B_t - B_{1/3}) & \text{if } s \in [1/3, 2/3] \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_s - B_{2/3}) & \text{if } s \in [2/3, 1] \end{cases}.$$
   (a) Show that I_t is a martingale.
   (b) Compute $\mathbb{E}[I_t^2]$.

4. **Increments of Martingales are not Correlated:** Let M_t be a martingale for the filtration $\mathcal{F}_t$. Use the properties of conditional expectation to show that for $t_1 < t_2 < t_3 < t_4$, we have
$$\mathbb{E}[(M_{t_2} - M_{t_1})(M_{t_4} - M_{t-3})] = 0.$$

5. **Not Everything is a Martingale** Show that a Gaussian process Y_t with the following covariance
$$C(Y_s, Y_t) = \frac{e^{-2(t-s)}}{2}(1 - e^{-2s})$$
is not a martingale.

Homework #7

#1

Let B_t be standard Brownian motion. Consider for $a,b>0$ the stopping time

$$\tau = \min_{t\geq 0}\{t : B_t \geq a \text{ or } B_t \leq -b\}.$$

(a) Show that $M_t = tB_t - \frac{1}{3}B_t^3$ is a martingale for the Brownian filtration.

(b) Use (a) to show that

$$\mathbb{E}[\tau B_\tau] = \frac{ab}{3}(a-b).$$

Solution:

(a)
$$\begin{aligned}\mathbb{E}[tB_t - \tfrac{1}{3}B_t^3 \mid \sigma(B_s)] &= t\,\mathbb{E}[B_t \mid \sigma(B_s)] - \tfrac{1}{3}\mathbb{E}[B_t^3 \mid \sigma(B_s)] \\ &= tB_s - \tfrac{1}{3}\mathbb{E}[(B_t - B_s + B_s)^3 \mid \sigma(B_s)] \\ &= tB_s - \tfrac{1}{3}\mathbb{E}[(B_t-B_s)^3 \mid \sigma(B_s)] - \tfrac{1}{3}[3(B_t-B_s)^2B_s \mid \sigma(B_s)] \\ &\quad - \tfrac{1}{3}\mathbb{E}[3(B_t-B_s)B_s^2 \mid \sigma(B_s)] - \tfrac{1}{3}\mathbb{E}[B_s^3 \mid \sigma(B_s)] \\ &= tB_s - \tfrac{1}{3}\mathbb{E}[(B_t-B_s)^3] - B_s\mathbb{E}[(B_t-B_s)^2] \\ &\quad - B_s\mathbb{E}[B_t-B_s] - \tfrac{1}{3}B_s^3 \\ &= tB_s - B_s(t-s) - \tfrac{1}{3}B_s^3 \\ &= sB_s - \tfrac{1}{3}B_s^3.\end{aligned}$$

Therefore, $tB_t - \frac{1}{3}B_t^3$ is a martingale.

(b) Applying Doob's optional stopping theorem we have that

$$\mathbb{E}[\tau B_\tau - \tfrac{1}{3}B_\tau^3] = \mathbb{E}[0B_0 - \tfrac{1}{3}0^3] = 0$$

However,

$$\begin{aligned}\mathbb{E}[\tau B_\tau - \tfrac{1}{3}B_\tau^3] &= \mathbb{E}[\tau B_\tau] - \tfrac{1}{3}\left(a^3P(B_\tau = a) - b^3P(B_\tau = -b)\right) \\ &= \mathbb{E}[\tau B_\tau] - \tfrac{1}{3}\left(\frac{a^3b}{a+b} - \frac{b^3a}{a+b}\right) \\ &= \mathbb{E}[\tau B_\tau] - \tfrac{1}{3}\left(ab(a^2-b^2)/a+b\right) \\ &= \mathbb{E}[\tau B_\tau] - \frac{ab}{3}(a-b).\end{aligned}$$

Therefore,

$$\mathbb{E}[\tau B_\tau] = \frac{ab}{3}(a-b).$$

$\blacksquare$

#2

Let B_t be a standard Brownian motion and let I_t be the stochastic process defined by

$$I_t = \begin{cases} 10B_t & t\in[0,1/3] \\ 10B_{1/3}+5(B_t-B_{1/3}), & t\in[1/3,2/3] \\ 10B_{1/3}+5(B_{2/3}-D_{1/3})+2(B_t-B_{2/3}), & t\in[2/3,1] \end{cases}$$

(a) Show that I_t is a martingale with respect to $\sigma(B_t)$.

(b) Compute $\mathbb{E}[I_t^2]$.

Solution:

(a) (i) If $t\in[0,1/3]$ we have that

$$\mathbb{E}[I_t|\sigma(B_s)] = \mathbb{E}[10B_t|\sigma(B_s)] = 10B_s = I_s$$

(ii) If $t\in[1/3,2/3]$ then

$$\begin{aligned}\mathbb{E}[I_t|\sigma(B_s)] &= \mathbb{E}[10B_{1/3}+5(B_t-B_{1/3})|\sigma(B_s)] \\ &= \mathbb{E}[10B_{1/3}|\sigma(B_s)]+5\mathbb{E}[B_t-B_{1/3}|\sigma(B_s)] \\ &= \begin{cases} 10B_s+5\mathbb{E}[B_t-B_{1/3}], & \text{if } s\in[0,1/3] \\ 10B_{1/3}+5B_s-5B_{1/3}, & \text{if } s\in[1/3,2/3] \end{cases} \\ &= \begin{cases} 10B_s, & \text{if } s\in[0,1/3] \\ 10B_{1/3}+5(B_s-B_{1/3}), & \text{if } s\in[1/3,2/3] \end{cases} \\ &= I_s\end{aligned}$$

(iii) If $t \in [2/3, 1]$ then

$$\mathbb{E}[I_t | \sigma(B_s)] = \mathbb{E}[10B_{1/3} + 5(B_{2/3} - B_{1/3}) + 2(B_t - B_{2/3}) | \sigma(B_s)]$$

$$= \begin{cases} 10B_s + 5\mathbb{E}[B_{2/3} - B_{1/3}] + 2\mathbb{E}[B_t - B_{2/3}] & \text{if } s \in [0, 1/3] \\ 10B_{1/3} + 5B_s - 5B_{1/3} + 2\mathbb{E}[B_t - B_{2/3}] & \text{if } s \in [1/3, 2/3] \\ 10B_{1/3} + 5(B_{2/3} - B_{1/3}) + 2B_s - 2B_{2/3} & \text{if } s \in [2/3, 1] \end{cases}$$

$$= \begin{cases} 10B_s & \text{if } s \in [0, 1/3] \\ 10B_{1/3} + 5(B_s - B_{1/3}) & \text{if } s \in [1/3, 2/3] \\ 10B_{1/3} + 5(B_{2/3} - B_{1/3} + 2(B_s - B_{2/3}) & \text{if } s \in [2/3, 1]. \end{cases}$$

$$= I_s,$$

Therefore, by items (i)-(iii) I_t is a martingale.

(b) Computing, and using independence of increments we have that

$$\mathbb{E}[I_t^2] = \begin{cases} \mathbb{E}[100B_t^2], & t \in [0, 1/3] \\ \mathbb{E}[100B_{1/3}^2] + 25\mathbb{E}[(B_t - B_{1/3})^2], & t \in [1/3, 2/3] \\ \mathbb{E}[100B_{1/3}^2] + 25\mathbb{E}[(B_{2/3} - B_{1/3})^2] + 4\mathbb{E}[(B_t - B_{2/3})^2], & t \in [2/3, 1] \end{cases}$$

$$= \begin{cases} 100t, & t \in [0, 1/3] \\ 100/3 + 25(t - 1/3), & t \in [1/3, 2/3] \\ 100/3 + 25/3 + 4(t - 2/3), & t \in [2/3, 1] \end{cases}$$

∎

#3

Let B_t be a standard Brownian motion and let I_t be defined by

$$I_t = \begin{cases} 0, & t \in [0, 1/3] \\ B_{1/3}(B_t - B_{1/3}), & t \in [1/3, 2/3] \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_t - B_{2/3}), & t \in [2/3, 1]. \end{cases}$$

(a) Show that I_t is a martingale.

(b) Compute $\mathbb{E}[I_t^2]$.

Solution:

(a) If $t \in [0, 1/3]$ then

$$\mathbb{E}[I_t \mid \sigma(B_s)] = \mathbb{E}[0 \mid \sigma(B_s)] = 0 = I_s.$$

If $t \in [1/3, 2/3]$ then

$$\begin{aligned}
\mathbb{E}[I_t \mid \sigma(B_s)] &= \mathbb{E}[B_{1/3}(B_t - B_{1/3}) \mid \sigma(B_s)] \\
&= \begin{cases} \mathbb{E}[(B_{1/3} - B_s + B_s)(B_t - B_{1/3}) \mid \sigma(B_s)], & s \in [0, 1/3] \\ B_{1/3}\,\mathbb{E}[B_t - B_{1/3} \mid \sigma(B_s)], & s \in [1/3, 2/3] \end{cases} \\
&= \begin{cases} \mathbb{E}[(B_{1/3} - B_s)(B_t - B_{1/3}) \mid \sigma(B_s)] + B_s\,\mathbb{E}[B_t - B_{1/3} \mid \sigma(B_s)], & s \in [0, 1/3] \\ B_{1/3}\,\mathbb{E}[B_t - B_s + B_s - B_{1/3} \mid \sigma(B_s)], & s \in [1/3, 2/3] \end{cases} \\
&= \begin{cases} \mathbb{E}[(B_{1/3} - B_s)(B_t - B_{1/3})] + B_s\,\mathbb{E}[B_t - B_{1/3}], & s \in [0, 1/3] \\ B_{1/3}(\mathbb{E}[B_t - B_s \mid \sigma(B_s)] + \mathbb{E}[B_s - B_{1/3} \mid \sigma(B_s)]), & s \in [1/3, 2/3] \end{cases} \\
&= \begin{cases} 0, & s \in [0, 1/3] \\ B_{1/3}\,\mathbb{E}[B_t - B_s] + B_{1/3}(B_s - B_{1/3}), & s \in [1/3, 2/3] \end{cases} \\
&= \begin{cases} 0, & s \in [0, 1/3] \\ B_{1/3}(B_s - B_{1/3}), & s \in [1/3, 2/3] \end{cases} \\
&= I_s.
\end{aligned}$$

If $t \in [2/3, 1]$ then

$$\mathbb{E}[I_t \mid \sigma(B_s)] = \mathbb{E}[B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_t - B_{2/3}) \mid \sigma(B_s)]$$

$$= \begin{cases} \mathbb{E}[(B_{1/3} - B_s + B_s)(B_{2/3} - B_{1/3}) + (B_{2/3} - B_s + B_s)(B_t - B_{2/3}) \mid \sigma(B_s)], & s \in [0, 1/3] \\ B_{1/3}\, \mathbb{E}[B_{2/3} - B_{1/3} \mid \sigma(B_s)] + \mathbb{E}[(B_{2/3} - B_s + B_s)(B_t - B_{2/3}) \mid \sigma(B_s)], & s \in [1/3, 2/3] \\ B_{1/3}\, \mathbb{E}[(B_{2/3} - B_{1/3}) \mid \sigma(B_s)] + B_{2/3}\, \mathbb{E}[B_t - B_{2/3} \mid \sigma(B_s)], & s \in [2/3, 1] \end{cases}$$

$$= \begin{cases} \mathbb{E}[(B_{1/3} - B_s)(B_{2/3} - B_{1/3}) \mid \sigma(B_s)] + B_s\, \mathbb{E}[B_{2/3} - B_{1/3} \mid \sigma(B_s)] + \mathbb{E}[(B_{2/3} - B_s)(B_t - B_{2/3}) \mid \sigma(B_s)] \\ \quad + B_s\, \mathbb{E}[B_t - B_{2/3} \mid \sigma(B_s)], & s \in [0, 1/3] \\ B_{1/3}\, \mathbb{E}[B_{2/3} - B_{1/3} \mid \sigma(B_s)] + (B_{2/3} - B_s)\, \mathbb{E}[B_t - B_{2/3} \mid \sigma(B_s)] + B_s\, \mathbb{E}[B_t - B_{2/3} \mid \sigma(B_s)], & s \in [1/3, 2/3] \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}\, \mathbb{E}[B_t - B_s + B_s - B_{2/3} \mid \sigma(B_s)], & s \in [2/3, 1] \end{cases}$$

$$= \begin{cases} \mathbb{E}[(B_{1/3} - B_s)(B_{2/3} - B_{1/3})] + B_s\, \mathbb{E}[B_{2/3} - B_{1/3}] + \mathbb{E}[(B_{2/3} - B_s)(B_t - B_{2/3})] + B_s\, \mathbb{E}[B_t - B_{2/3}], & s \in [0, \\ B_{1/3}(B_s - B_{1/3}) + (B_{2/3} - B_s)\, \mathbb{E}[B_t - B_{2/3}] + B_s\, \mathbb{E}[B_t - B_{2/3}], & s \in [1/3, 2/3] \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_s - B_{2/3}), & s \in [2/3, 1]. \end{cases}$$

$$= \begin{cases} 0, & s \in [0, 1/3] \\ B_{1/3}(B_s - B_{1/3}), & s \in [1/3, 2/3] \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_s - B_{2/3}), & s \in [2/3, 1]. \end{cases}$$

$$= I_s.$$

(b) $$\mathbb{E}[I_t^2] = \begin{cases} \mathbb{E}[0], & t \in [0, 1/3] \\ \mathbb{E}[B_{1/3}^2 (B_t - B_{1/3})^2], & t \in [1/3, 2/3] \\ \mathbb{E}[(B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_t - B_{2/3}))^2], & t \in [2/3, 1] \end{cases}$$

$$= \begin{cases} 0, & t \in [0, 1/3] \\ \mathbb{E}[B_{1/3}^2] \cdot \mathbb{E}[(B_t - B_{1/3})^2], & t \in [1/3, 2/3] \\ \mathbb{E}[B_{1/3}^2] \cdot \mathbb{E}[(B_{2/3} - B_{1/3})^2] + \mathbb{E}[B_{2/3}^2]\, \mathbb{E}[(B_t - B_{2/3})^2], & t \in [2/3, 1] \end{cases}$$

$$= \begin{cases} 0, & t \in [0, 1/3] \\ 1/3\,(t - 1/3), & t \in [1/3, 2/3] \\ 1/3\,(1/3) + 2/3\,(t - 2/3), & t \in [2/3, 1] \end{cases}$$

$\blacksquare$

#4

Let M_t be a martingale for the filtration $\mathcal{F}_t$. Use properties of conditional expectation to show that

$$\mathbb{E}[(M_{t_2}-M_{t_1})(M_{t_4}-M_{t_3})]=0$$

Solution:

$$\begin{aligned}
\mathbb{E}[(M_{t_2}-M_{t_1})(M_{t_4}-M_{t_3})] &= \mathbb{E}[\mathbb{E}[(M_{t_2}-M_{t_1})(M_{t_4}-M_{t_3}) \mid \mathcal{F}_{t_2}]] \\
&= \mathbb{E}[(M_{t_2}-M_{t_1})\mathbb{E}[M_{t_4}-M_{t_3} \mid \mathcal{F}_{t_2}]] \\
&= \mathbb{E}[(M_{t_2}-M_{t_1})(M_{t_2}-M_{t_2})] \\
&= \mathbb{E}[0] \\
&= 0.
\end{aligned}$$

#5

Show that a Gaussian process Y_t with the following covariance

$$C(Y_s, Y_t) = \frac{e^{-2(t-s)}}{2}(1-e^{-2s})$$

is not a martingale.

Solution:

Let $t_1 < t_2 \leq t_3 < t_4$. Then,

$$\begin{aligned}
Cov((Y_{t_2}-Y_{t_1}),(Y_{t_4}-Y_{t_3})) &= Cov(Y_{t_2},Y_{t_4}) - Cov(Y_{t_1},Y_{t_4}) \\
&\quad - (Cov(Y_{t_2},t_3) + Cov(Y_{t_1},Y_{t_3}))
\end{aligned}$$

Assuming $t_1=0$, $t_2=\ln(2)$, $t_3=\ln(3)$, $t_4=\ln(4)$ we have

$$\begin{aligned}
Cov(Y_{t_2},Y_{t_4}) &= e^{-2(\ln(4)-\ln(2))}/2\,(1-e^{-2\ln(2)}) \\
&= 1/8\,(3/4) \\
&= 3/32
\end{aligned}$$

$$\mathrm{Cov}(Y_{t_1}, Y_{t_4}) = e^{-2\ln(4)/2}$$
$$= 1/32$$
$$\mathrm{Cov}(Y_{t_2}, Y_{t_3}) = e^{-2(\ln(3)-\ln(2))/2}(1-e^{-2\ln(2)})$$
$$= 4/18(1-1/4)$$
$$= 3/18$$
$$\mathrm{Cov}(Y_{t_1}, Y_{t_3}) = e^{-2\ln(3)/2}(\ldots$$
$$= 1/18$$

Therefore,

$$\mathrm{Cov}((Y_{t_2}-Y_{t_1})(Y_{t_4}-Y_{t_3})) = 3/32 - 1/32 - 3/18 + 1/18$$
$$= 1/16 - 1/9$$
$$\neq 0.$$

Consequently, since $\mathbb{E}[(M_{t_2}+M_{t_1})(M_{t_4}-M_{t_3})] \neq 0$ it follows that M_t is not a martingale.

$\blacksquare$