MTH 383/683: Homework #8

Due Date: November 20, 2023

1 Problems for Everyone

1. **Practice on Ito Integrals:** Consider the four processes:

$$X_t = \int_0^t (1-s)dB_s, \qquad Y_t = \int_0^t (1+s)dB_s, \qquad Z_t = \int_0^t \sin(s)dB_s, \qquad W_t = \int_0^t \cos(s)dB_s.$$

   (a) Explain why each of these processes are Gaussian.

   (b) Find the mean and covariance for each of these processes.

   (c) Determine the probability densities for each of these processes.

   (d) For which time, if any, do we have that X_t and Y_t are uncorrelated? Are X_t and Y_t independent at these times?

   (e) Determine the covariance matrix for the Gaussian random variable $(Z_{\pi/2}, Z_\pi)$.

   (f) Write down the double integral for the probability $P(Z_{\pi/2} > 1, Z_\pi > 1)$.

   (g) Determine for which times the processes Z_t and W_t are indpendent.

2. **Integration By Parts for some Ito Integrals:** Let g be a smooth function and B_t a standard Brownian motion.

   (a) Use Ito's formula to prove that for any $t \geq 0$

$$\int_0^t g(s)dB_s = g(t)B_t - \int_0^t B_s g'(s)ds.$$

   (b) Show that the process given by

$$X_t = t^2 B_t - 2\int_0^t sB_s ds$$

   is Gaussian. Find its mean and covariance.

3. **Some Practice with Ito's Formula:** Show that:

   (a) $\displaystyle\int_0^t B_s^3 dB_s = \frac{1}{4}B_t^4 - \frac{3}{2}\int_0^t B_s^2 ds$

   (b) $\displaystyle\int_0^t B_s^4 dB_s = \frac{1}{5}B_t^5 - 2\int_0^t B_s^3 ds$

   (c) $\displaystyle\int_0^t B_s^{n-1} dB_s = \frac{1}{n}B_t^n - \frac{n-1}{2}\int_0^t B_s^{n-2} ds$, where $n \in \mathbb{N}$ and $n > 2$.

4. **Some Practice with Ito's Formula and Compensators:** Let B_t be a standard Brownian motion and consider the three processes:

$$X_t = \int_0^t \cos(s) dB_s, \qquad Y_t = B_t^4, \qquad Z_t = (B_t + t)\exp(-B_t - t/2).$$

(a) Determine if each of these processes is a martingale, if not find a compensator for it.

(b) Find the mean, the variance, and the covariance of each of these processes.

(c) Determine if each of these processes are Gaussian.

5. **Gaussian Moments Using Ito:** Let B_t be a Brownian motion. Use Ito's formula to show that for $k \in \mathbb{N}$

$$\mathbb{E}[B_t^k] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_s^{k-2}]ds.$$

Conclude from this that $\mathbb{E}[B_t^4] = 3t^2$ and $\mathbb{E}[B_t^6] = 15t^3$.

Homework #8

#1

Consider the four processes:

$$X_t = \int_0^1 (1-s)\,dB_s,\quad Y_t = \int_0^1 (1+s)\,dB_s,\quad Z_t = \int_0^t \sin(s)\,dB_s,\quad W_t = \int_0^t \cos(s)\,dB_s$$

(a) Explain why each of these processes are Gaussian

(b) Find the mean and covariance for each of these processes.

(c) Determe the probability density for each process.

(d) For which time, if any, do we have that X_t, Y_t are uncorrelated? Are X_t, Y_t independent at these times?

(e) Determine the covariance matrix for the Gaussian random vector $(Z_{\pi/2}, Z_\pi)$.

(f) Write down the double integral for the probability $P(Z_{\pi/2} > 1, Z_\pi > 1)$

(g) Determine for which times the process Z_t and W_t are independent.

Solution:

(b) $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = \mathbb{E}[Z_t] = \mathbb{E}[W_t] = 0$. Now, assuming $s \leq t$,

- $\mathbb{E}[X_t X_s] = \int_0^s (1-u)^2 du = s - s^2 + \frac{1}{3}s^3.$
- $\mathbb{E}[Y_t Y_s] = \int_0^s (1+u)^2 du = s + s^2 + \frac{1}{3}s^3.$
- $\mathbb{E}[Z_t Z_s] = \int_0^s \sin^2(u) du$

$$= \int_0^s \frac{1-\cos(2u)}{2} du$$

$$= \frac{1}{2}s - \frac{1}{4}\sin(2s).$$

- $\mathbb{E}[W_t W_s] = \int_0^s \cos^2(u)\, du$

$$= \int_0^s \frac{1+\cos(2u)}{2} du$$

$$= \frac{1}{2}s + \frac{1}{4}\sin(2u).$$

(c). We have that

$$X_t \sim \frac{1}{\sqrt{2\pi t(1-t+\frac{1}{3}t^2)}} \exp\left(\frac{-x^2}{2t(1-t+\frac{1}{3}t^2)}\right)$$

$$Y_t \sim \frac{1}{\sqrt{2\pi t(1+t+\frac{1}{3}t^2)}} \exp\left(\frac{-y^2}{2t(1+t+\frac{1}{3}t^2)}\right)$$

$$Z_t \sim \frac{1}{\sqrt{2\pi(\frac{1}{2}t-\frac{1}{4}\sin(2t))}} \exp\left(\frac{-z^2}{2(\frac{1}{2}t-\frac{1}{4}\sin(2t))}\right)$$

$$W_t \sim \frac{1}{\sqrt{2\pi(\frac{1}{2}t+\frac{1}{4}\sin(2t))}} \exp\left(\frac{-w^2}{2(\frac{1}{2}t+\frac{1}{4}\sin(2t))}\right)$$

(d) $$\mathbb{E}[X_t Y_t] = \int_0^t (1-s)(1+s)\,ds = \int_0^t (1-s^2)\,ds = t\left(1-\frac{1}{3}t^2\right)$$

Therefore, these processes are independent when $t = \sqrt{3}$.

(e). $\mathbb{E}[Z_{\pi/2} Z_\pi] = \frac{1}{4}\pi$

$\mathbb{E}[Z_{\pi/2} Z_{\pi/2}] = \frac{1}{4}\pi$

$\mathbb{E}[Z_\pi Z_\pi] = \frac{1}{2}\pi$

Therefore, the covariance matrix is given by

$$C = \begin{bmatrix} \frac{1}{4}\pi & \frac{1}{4}\pi \\ \frac{1}{4}\pi & \frac{1}{2}\pi \end{bmatrix} \Rightarrow \det(C) = \frac{1}{8}\pi^2 - \frac{1}{16}\pi^2 = \frac{1}{16}\pi^2.$$

(f) $$P(Z_{\pi/2} > 1, Z_\pi > 1) = \frac{1}{2\pi \cdot \frac{1}{16}\pi^2} \int_1^\infty \int_1^\infty \exp\left(-\frac{[z_1\ z_2]\, C^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}{2}\right) dz_1\, dz_2.$$

(g) $$\mathrm{Cov}(Z_t, W_t) = \int_0^t \sin(u)\cos(u)\,du = \int_0^t \frac{d}{du}\frac{1}{2}\sin^2(u)\,du = \frac{1}{2}\sin^2(t).$$

Therefore, Z_t, W_t are independent when $t = n\pi$ for $n \in \mathbb{N}$. ∎

#2.

Let g be a smooth function and B_t a standard Brownian motion.

(a) Use Ito's formula to prove that for any $t \geq 0$

$$\int_0^t g(s)dB_s = g(t)B_t - \int_0^t B_s g'(s)ds.$$

(b) Show that the process given by

$$X_t = t^2 B_t - 2\int_0^t s B_s ds.$$

is Gaussian. Find its mean and covariance.

Solution:

(a) By Ito's formula we have

$$g(t)B_t = \int_0^t \frac{\partial}{\partial s}(g(s)B_s)ds + \int_0^t \frac{\partial}{\partial B}(g(s)B_s)dB + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial B^2}(g(s)B_s)ds.$$

$$= \int_0^t g'(s)B_s ds + \int_0^t g(s)dB$$

$$\Rightarrow \int_0^t g(s)dB_s = g(t)B_t - \int_0^t g'(s)B_s ds.$$

(b) By part (a) we have that

$$t^2 B_t - 2\int_0^t s B_s ds = \int_0^t s^2 dB_s$$

and thus is Gaussian. Moreover

$$\mathbb{E}\left[\int_0^t s^2 dB_s\right] = 0$$

$$\mathbb{E}\left[\left(\int_0^t s^2 dB_s\right)^2\right] = \int_0^t \mathbb{E}[s^4]ds$$

$$= \int_0^t s^4 ds$$

$$= \frac{1}{5}t^5.$$

∎

#3

Show that

(a) $\int_0^t B_s^3 \, dB_s = \frac{1}{4} B_t^4 - \frac{3}{2} \int_0^t B_s^2 \, ds.$

(b) $\int_0^t B_s^4 \, dB_s = \frac{1}{5} B_t^5 - 2 \int_0^t B_s^3 \, ds$

(c) $\int_0^t B_s^{n-1} \, dB_s = \frac{1}{n} B_t^n - \frac{n-1}{2} \int_0^t B_s^{n-2} \, ds$, where $n \in \mathbb{N}$ and $n \geq 2$.

Solution:

(a) By Ito's formula

$$B_t^4 = \int_0^t 4 B_s^3 \, dB + \frac{1}{2} \int_0^t 12 B_s^2 \, ds$$

$$\Rightarrow \frac{1}{4} B_t^4 - \frac{3}{2} \int_0^t B_s^2 \, ds = \int_0^t B_s^3 \, dB.$$

(b) By Ito's formula

$$B_t^5 = \int_0^t 5 B_s^4 \, dB + \frac{1}{2} \int_0^t 20 B_s^3 \, ds$$

$$\Rightarrow \frac{1}{5} B_t^5 - 2 \int_0^t B_s^3 \, ds = \int_0^t B_s^4 \, dB.$$

(c) By Ito's formula

$$B_t^n = \int_0^t n B_s^{n-1} \, dB + \frac{1}{2} \int_0^t n(n-1) B_s^{n-2} \, ds$$

$$\Rightarrow \frac{1}{n} B_t^n - \frac{n-1}{2} \int_0^t B_s^{n-2} \, ds = \int_0^t B_s^{n-1} \, dB.$$

$\blacksquare$

#4

Let B_t be a standard Brownian motion and consider the three processes:

$$X_t = \int_0^t \cos(s) \, dB_s, \quad Y_t = B_t^4, \quad Z_t = (B_t + t) \exp(-B_t - t/2).$$

(a) Determine if each of these processes is a martingale, if not find a compensator for it.

(b) Find the mean, variance, and covariance of these processes.

(c) Determine if each of these processes are Gaussian.

(a) Since X_t is an Ito integral it is a martingale. Since, $\mathbb{E}[B_t^4] = 3t^2 \neq 0 = \mathbb{E}[B_0^4]$ it follows that B_t^4 is not a martingale. By Ito's formula we have that

$$
\begin{aligned}
(B_t + t)\exp(-B_t - t/2) &= \int_0^t \left(1 - \tfrac{1}{2}(B_t + t)\right)\exp(-B_t - t/2)\,dt \\
&\quad + \int_0^t \left(1 - (B_t + t)\right)\exp(-B_t - t/2)\,dB \\
&\quad + \tfrac{1}{2}\int_0^t \left(-1 - 1 + (B_t + t)\right)\exp(-B_t - t/2)\,dt \\
&= \int_0^1 (1 - B_t - t)\exp(-B_t - t/2)\,dB.
\end{aligned}
$$

Therefore, since $(B_t + t)\exp(-B_t - t/2)$ is an Ito integral it is a martingale.

Since $B_t^4 = \int_0^t 4B_t^3\,dB + 6\int_0^t B_t^2\,dt$ it follows that $B_t^4 - 6\int_0^t B_t^2\,dt$ is a martingale. Therefore, the compensator of B_t^4 is $6\int_0^t B_t^2\,dt$.

(b) For X_t:

$$
\mathbb{E}[X_t] = 0
$$

$$
\begin{aligned}
\mathbb{E}[X_t X_{t'}] &= \int_0^{\min\{t,t'\}} \cos^2(s)\,ds \\
&= \int_0^{\min\{t,t'\}} \tfrac{1}{2}(1 + \cos(2s))\,ds \\
&= \tfrac{1}{2}\min\{t,t'\} + \tfrac{1}{4}\sin(2\min\{t,t'\}).
\end{aligned}
$$

$$
\Rightarrow \mathbb{E}[X_t^2] = \tfrac{1}{2}t + \tfrac{1}{4}\sin(2t).
$$

For Y_t:

$$
\mathbb{E}[Y_t] = \mathbb{E}[B_t^4] = 3t^2
$$

$$
\begin{aligned}
\mathbb{E}[Y_t^2] - \mathbb{E}[Y_t]^2 &= 7\cdot 5\cdot 3\,t^4 - 9t^4 \\
&= 3(35 - 3)t^4 \\
&= 96t^4.
\end{aligned}
$$

Assuming $t < t'$ we have

$$
\begin{aligned}
\mathbb{E}[Y_t Y_{t'}] &= \mathbb{E}[B_t^4 B_{t'}^4] \\
&= \mathbb{E}[B_t^4 (B_{t'} - B_t + B_t)^4] \\
&= \mathbb{E}[B_t^4 ((B_{t'} - B_t)^4 + 4(B_{t'} - B_t)^3 B_t + 6(B_{t'} - B_t)^2 B_t^2 + 4(B_{t'} - B_t) B_t^3 + B_t^4)] \\
&= \mathbb{E}[B_t^4]\mathbb{E}[(B_{t'} - B_t)^4] + 4\mathbb{E}[(B_{t'} - B_t)^3]\mathbb{E}[B_t^5] + 6\mathbb{E}[(B_{t'} - B_t)^2]\mathbb{E}[B_t^6] \\
&\quad + 4\mathbb{E}[(B_{t'} - B_t)]\mathbb{E}[B_t^7] + \mathbb{E}[B_t^8] \\
&= 9t^2(t'-t)^2 + 0 + 6(t'-t)15t^3 + 0 + 7\cdot 5\cdot 3t^4
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \operatorname{Cov}(Y_t, Y_{t'}) &= \mathbb{E}[Y_t Y_{t'}] - \mathbb{E}[Y_t]\mathbb{E}[Y_{t'}] \\
&= 9t^2(t'-t)^2 + 6\cdot 15(t'-t)t^3 + 7\cdot 5\cdot 3t^4 - 9t^2 t'^2 \\
&= 3t^2(3(t'-t)^2 + 2\cdot 15(t'-t)t + 7\cdot 5t^2 - 3t'^2) \\
&= 3t^2(3(t'^2 - 2t't + t^2) + 30(t'-t)t + 35t^2 - 3t'^2) \\
&= 3t^2(3t'^2 - 6t't + 3t^2 + 35t^2 - 3t'^2) \\
&= 3t^3(38t - 6t')
\end{aligned}
$$

(c) X_t is Gaussian since it is a sum of Gaussians. Y_t is given by

$$Y_t = \int_0^t 4B^3 dB + \frac{1}{2}\int_0^t 12B^2 dt$$

and is thus not a Gaussian. Finally, since

$$Z_t = \int_0^t (1 - B_t - t)\exp(-B_t - t/2)\, dB$$

it is not Gaussian.

■

#5

Let B_t be a Brownian motion. Use Itô's formula to show that for $k \in \mathbb{N}$

$$\mathbb{E}[B_t^k] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_s^{k-2}]ds.$$

Conclude from this that $\mathbb{E}[B_t^4] = 3t^2$ and $\mathbb{E}[B_t^6] = 15t^3$.

Solution:

By Itô's formula

$$B_t^k = \int_0^t k B_t^{k-1} dB + \frac{1}{2}\int_0^t k(k-1) B_t^{k-2} dt$$

$$\Rightarrow \mathbb{E}[B_t^k] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_t^{k-2}]dt$$

Therefore,

$$\mathbb{E}[B_t^4] = \frac{1}{2}4\cdot 3\int_0^t \mathbb{E}[B_t^2]dt$$
$$= \frac{1}{2}4\cdot 3\int_0^t t\,dt$$
$$= 3t^2.$$

$$\mathbb{E}[B_t^6] = \frac{1}{2}6\cdot 5\int_0^t \mathbb{E}[B_t^4]dt$$
$$= 3\cdot 5\int_0^t 3t^2 dt$$
$$= 15t^3$$

■