MTH 383/683: Homework #9

Due Date: December 01, 2023

1 Problems for Everyone

1. **Exercise on Ito's Formula:** Consider the process

$$X_t = \exp(tB_t).$$

   (a) Explain briefly why $Z_t = tB_t$ is a Gaussian random variable and find its mean and variance.

   (b) Find the mean and variance of X_t.

   (c) Use Ito's formula to write Z_t in terms of an Ito integral and a Riemann integral.

   (d) Find a compensator C_t so that $X_t - C_t$ is a martingale.

   (e) Show that the covariance between B_t and $\int_0^t e^{sB_s} dB_s$ at time t is

$$\operatorname{Cov}\left(B_t, \int_0^t e^{sB_s} dB_s\right) = \int_0^t e^{s^3/2} ds.$$

2. **Martingales and Ito's Formula:** Prove that

$$Y_t = e^{t/2}\cos(B_t)$$

is a martingale.

3. **Random Time Blowup:** Consider the following stochastic differential equation:

$$dX = -\frac{1}{2}e^{-2X}dt + e^{-X}dB$$
$$X(0) = x_0.$$

   (a) Use the substitution $X = u(B)$ to solve this stochastic differential equation.

   (b) Show that the solution diverges at a finite but random time.

4. **Solving an SDE:** Solve the following stochastic differential equations

   (a) $dX = -Xdt + e^{-t}dB$, $X(0) = 0$

   (b) $dX = -X/(1+t)dt + 1/(1+t)dB$, $X(0) = 1$

   (c) $dX = -X/(1-t)dt + dB$, $X(0) = 0$

   (d) $dX = XBdt + dB$, $X(0) = x_0$.

5. **Brownian Motion on the Unit Circle:** Consider the following system of stochastic differential equations

$$\begin{cases} dX &= -\frac{1}{2}X dt - Y dB \\ dY &= -\frac{1}{2}Y dt + X dB \end{cases}.$$

(a) Show that for any solution to this stochastic differential equation, $X^2 + Y^2$ is constant in time.

(b) Show that $X = (\cos(B), \sin(B))$ solves this system.

Homework #9

#1

Consider the process

$$X_t = \exp(tB_t)$$

(b) Find the mean and variance of X_t.

(c) Use Ito's formula to write Z_t in terms of an Ito integral and a Riemann integral.

(d) Find a compensator C_t so that $X_t - C_t$ is a martingale.

(e) Show that the covariance between B_t and $\int_0^t e^{sB_s} dB_s$ at time t is

$$\mathrm{Cov}\left(B_t, \int_0^t e^{sB_s} dB_s\right) = \int_0^t e^{s^3/2} ds.$$

Solution:

(b) $\mathbb{E}[X_t] = \mathbb{E}[e^{tB_t}]$

$= e^{t^3/2}$

$\mathbb{E}[X_t^2] = \mathbb{E}[e^{2tB_t}]$

$= e^{2t^3}$

Therefore,

$\mathrm{Var}(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2$

$= e^{2t^3} - e^{t^3}$

$= e^{t^3}$

(c) $Z_t = tB_t$

$\Rightarrow Z_t = \int_0^t s\, dB_s + \int_0^t B_s ds$

(d) By Ito's formula

$$e^{tB} - 1 = \int_0^t Be^{sB} ds + \int_0^t se^{sB} dB + \frac{1}{2}\int_0^t s^2 e^{sB} ds$$

$$\Rightarrow e^{tB} - \int_0^t Be^{sB} ds - \frac{1}{2}\int_0^t s^2 e^{sB} ds = 1 + \int_0^t se^{sB} dB.$$

Therefore,

$$e^{tB} - \int_0^t B e^{sB} ds - \frac{1}{2}\int_0^t s^2 e^{sB} ds$$

is a martingale.

(e) Computing, we have that

$$\begin{aligned} Cov\left(B_t, \int_0^t e^{sB_s} dB_s\right) &= Cov\left(\int_0^t dB_s, \int_0^t e^{sB_s} dB_s\right) \\ &= \int_0^t \mathbb{E}[e^{sB_s}] ds \\ &= \int_0^t e^{s^3/2} ds \end{aligned}$$

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#2.

Prove that $Y_t = e^{t/2}\cos(B_t)$ is a martingale.

Solution:

By Ito's formula we have that

$$e^{t/2}\cos(B_t) - 1 = \int_0^t \frac{1}{2} e^{s/2}\cos(B) ds - \int_0^t e^{s/2}\sin(B) dB - \frac{1}{2}\int_0^t e^{s/2}\cos(B) ds$$

$$\Rightarrow e^{t/2}\cos(B) = 1 - \int_0^t e^{s/2}\sin(B) dB$$

and thus is a martingale.

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#3

Consider the following stochastic differential equation

$$dX = -\frac{1}{2} e^{-2X} dt + e^{-X} dB$$

$$X(0) = x_0$$

(a) Use the substitution $X = u(B)$ to solve this SDE.

(b) Show that the solution diverges at a finite but random time.

Solution:

(a) Assuming $X = u(B)$ we have that

$$dX = u'(B)\,dB + \tfrac{1}{2}u''(B)\,dt$$

$$\Rightarrow u'(B) = e^{-u}$$

$$\Rightarrow e^{u}\frac{du}{dB} = 1$$

$$\Rightarrow \int_{x_0}^{X} e^{u}\,du = \int_0^{B} dB$$

$$\Rightarrow e^{X} - e^{x_0} = B$$

$$\Rightarrow X = \ln(B + e^{x_0}).$$

(b) The solution diverges at the time when $B = -e^{x_0}$

$\blacksquare$

#4

Solve the following stochastic differential equations.

(a) $dX = -X\,dt + e^{-t}\,dB,\ X(0) = 0$

(b) $dX = -X/(1+t)\,dt + 1/(1+t)\,dB,\ X(0) = 1$

(c) $dX = -X/(1-t)\,dt + dB,\ X(0) = 0$

(d) $dX = XB\,dt + dB,\ X(0) = x_0.$

Solution:

(a) $dX + X\,dt = e^{-t}\,dB$

$$\Rightarrow e^{g(t)}dX + e^{g(t)}X\,dt = e^{g(t)}e^{-t}\,dB$$

Therefore, if $d(e^{g(t)}X) = e^{g(t)}dX + g'(t)e^{g(t)}X\,dt$ then $g'(t) = 1$. Consequently, $g(t) = t$

$$\Rightarrow d(e^{t}X) = dB$$

$$\Rightarrow e^{t}X = B$$

$$\Rightarrow X = Be^{-t}$$

(b) $dX + \frac{1}{(1+t)} X\,dt = \frac{1}{1+t} dB$

$$\Rightarrow e^{g(t)} dX + \frac{e^{g(t)}}{1+t} X\,dt = \frac{e^{g(t)}}{1+t} dB$$

Therefore, if $d(e^{g(t)}X) = e^{g(t)} dX + e^{g(t)}(1+t)^{-1} X\,dt$ it follows that

$$g'(t) = \frac{1}{1+t}$$

$$\Rightarrow g(t) = \ln(1+t).$$

$$\Rightarrow d((1+t)X) = dB$$

$$\Rightarrow (1+t)X - 1 = B$$

$$\Rightarrow X = \frac{1}{1+t} + \frac{B}{1+t}.$$

(c) $dX + \frac{X}{1-t} dt = dB$

$$\Rightarrow d((1-t)^{-1}X) = (1-t)^{-1} dB$$

$$\Rightarrow (1-t)^{-1} X = \int_0^t (1-s)^{-1} dB_s$$

$$\Rightarrow X = 1-t \int_0^t (1-s)^{-1} dB_s.$$

(d) $dX - XB\,dt = dB$

$$\Rightarrow d(e^{-\int_0^t B_s ds} X) = e^{-\int_0^t B_s ds} dB$$

$$\Rightarrow e^{-\int_0^t B_s ds} X - X_0 = \int_0^t e^{-\int_0^u B_s ds} dB_u$$

$$\Rightarrow X = x_0 e^{\int_0^t B_s ds} + e^{\int_0^t B_s ds} \int_0^t e^{-\int_0^u B_s ds} dB_u$$

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#5

Consider the following system of stochastic differential equations

$$\begin{cases} dX = -\frac{1}{2}X\,dt - Y\,dB \\ dY = -\frac{1}{2}Y\,dt + X\,dB \end{cases}$$

(a) Show that for any solution to this stochastic differential equation, X^2+Y^2 is constant in time.

(b) Show that $X = (\cos(B), \sin(B))$ solves this system.

Solution:

(a)
$$\begin{aligned} d(X^2+Y^2) &= 2X\,dX + 2Y\,dY + \tfrac{1}{2}(2dX^2 + 2dY^2) \\ &= 2X(-\tfrac{1}{2}X\,dt - Y\,dB) + 2Y(-\tfrac{1}{2}Y\,dt + X\,dB) \\ &\quad + Y^2\,dt + X^2\,dB \\ &= 0. \end{aligned}$$

(b) Letting $X = \cos(B)$, $Y = \sin(B)$ we have that

$$\begin{cases} dX = -\sin(B)\,dB - \frac{1}{2}\cos(B)\,dt \\ dY = \cos(B)\,dB - \frac{1}{2}\sin(B)\,dt \end{cases}$$

$$\begin{cases} -\frac{1}{2}X\,dt - Y\,dB = -\frac{1}{2}\cos(B)\,dt - \sin(B)\,dB \\ -\frac{1}{2}Y\,dt + X\,dB = -\frac{1}{2}\sin(B)\,dt + \cos(B)\,dB \end{cases}$$

and thus $(\cos(B), \sin(B))$ is a solution. ∎