MTH 383/683: Homework #1

Due Date: September 08, 2023

1 Problems for Everyone

1. **Basic properties of probability**. Let P be a probability on Ω. Use the basic properties of probability to prove the following

   (a) *Finite Additivity*: If A, B are disjoint events then $P(A \cup B) = P(A) + P(B)$.

   (b) For any event A, $P(A^c) = 1 - P(A)$.

   (c) For any events A, B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   (d) *Monotinicity*: If $A \subset B$, $P(A) \leq P(B)$.

2. **Distribution as a probability on** $\mathbb{R}$. Let ρ_X be the distribution of a random variable X on some probability space $(\Omega, \mathcal{F}, P)$. Show that ρ_X has the properties of a probability distribution on $\mathbb{R}$.

3. **Distribution of an indicator function**. Let $(\Omega, \mathcal{F}, P)$ be a probability space and A and event in $\mathcal{F}$ with $0 < P(A) < 1$. What is the distribution of the random variable $\mathbb{1}_A$?

4. **Constructing a random variable from another one**. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1, 1]$. Consider $Y = X^2$.

   (a) Find the CDF of Y and plot its graph.

   (b) Find the PDF of Y and plot its graph.

5. **Memory loss property**. Let Y be an exponential random variable with parameter $\lambda > 0$. Show that for any $s, t > 0$

$$P(Y > t + s | Y > s) = P(Y > t).$$

   Recall that the conditional probability of A given the event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

6. **Independence** Recall that two events A and B in some probability space $(\Omega, \mathcal{F}, P)$ are independent if

$$P(A \cap B) = P(A)P(B).$$

   Consequently, if A and B are independent it follows that $P(A|B) = P(A)$ as expected.

   (a) Prove that if A and B are independent than A^c and B are also independent.

   (b) Prove that if A and B are independent then A^c and B^c are also independent.

   (c) Prove that if A and B are independent then $P(A \cup B) = 1 - (1 - P(A))(1 - P(B))$.

2 Graduate Problems (undergraduates can complete for extra credit but your homework score cannot go above 10 points)

1. Suppose $A_1, A_2, \ldots$ are events in a probability space $(\Omega, \mathcal{F}, P)$.

   (a) Prove that

   $$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{\omega \in \Omega : \omega \text{ belongs to infinitely many } A_n\}.$$

   The event $\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ is called "A_n infinitely often" and is abbreviated "A_n i.o.".

   (b) Prove that if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then

   $$P(A_n \text{ i.o.}) = 0.$$

   (c) $A_1, A_2, \ldots$ are called mutually independent if every A_i is independent of any intersection of the other A_j for $j \neq i$, that is, for every finite subsequence A_{j_k} of events

   $$P\left(\bigcap_k A_{j_k}\right) = \prod_k P(A_{j_k}).$$

   Prove that if $A_1, A_2, \ldots$ are mutually independent events then

   $$P\left(\bigcup_{k=n}^{\infty} A_k\right) = 1 - \prod_{k=n}^{\infty} \left(1 - P(A_k)\right).$$

   (d) Prove that for all $x \in \mathbb{R}$, $1 - x \leq e^{-x}$ and use this to prove that for mutually independent events $A_1, A_2, \ldots$ it follows that

   $$P\left(\bigcup_{k=n}^{\infty} A_k\right) \geq 1 - \prod_{k=n}^{\infty} e^{-P(A_k)} = 1 - e^{\sum_{k=n}^{\infty} A_k}.$$

   (e) If $A_1, A_2, \ldots$ are mutually independent events, prove that if $\sum_{n=1}^{\infty} P(A_n) = \infty$ then

   $$P(A_n \text{ i.o.}) = 1.$$

   (f) Suppose that the events $A_1, A_2, \ldots$ are mutually independent with

   $$P\left(\bigcup_n A_n\right) = 1 \text{ and } P(A_n) < 1$$

   for each n. Prove that $P(A_n \text{ i.o.}) = 1$.

2. Let Ω be any set and $\mathcal{A}$ any collection of subsets of Ω. Show that there exists a unique smallest σ-algebra $\mathcal{F}$ of subsets of Ω containing $\mathcal{A}$. We call $\mathcal{F}$ the σ-algebra generated by $\mathcal{A}$. **Hint:** Consider the intersection of all σ-algebras containing $\mathcal{A}$.