MTH 383/683: Homework #5

Due Date: October 20, 2023

1 Problems for Everyone

1. **Triangle Inequality** Let X, Y be two random variables in $L^2(\Omega, \mathcal{F}, P)$. Prove the triangle inequality
$$\|X + Y\| \leq \|X\| + \|Y\|.$$
Hint: Write out the norms as expectations, expand, and use the Cauchy-Schwarz inequality.
Use this result to prove the more general inequality: for $X_1, \ldots X_n \in L^2(\Omega, \mathcal{F}, P)$ we have
$$\|X_1 + \ldots X_n\| \leq \sum_{j=1}^{n} \|X_i\|.$$

2. **Integrable Random Variables** Recall that $L^1(\Omega, \mathcal{F}, P)$ is the set of all random variables for which $\mathbb{E}[|X|] < \infty$.
   (a) Prove that $L^1(\Omega, \mathcal{F}, P)$ is a linear subspace of the vector space of random variables.
   (b) Verify that $\|X\|_1 = \mathbb{E}[|X|]$ is a norm for L^1. That is prove the following
      i. For all $a \in \mathbb{R}$ and $X \in L^1$ that $\|aX\|_1 = |a|\|X\|_{L^1}$.
      ii. For all $X \in L^1$, $\|X\|_{L^1} = 0$ if and only if $X = 0$.
      iii. For all $X, Y \in L^1$, $\|X + Y\| \leq \|X\| + \|Y\|$.

3. **Brownian Moments** Let B_t be standard Brownian motion. Compute the following moments:
   (a) $\mathbb{E}[B_t^6]$
   (b) $\mathbb{E}[(B_{t_2} - B_{t_1})(B_{t_3} - B_{t_2})]$ if $t_1 < t_2 < t_3$.
   (c) $\mathbb{E}[B_s^2 B_t^2]$ if $s < t$
   (d) $\mathbb{E}[B_s B_t^3]$ if $s < t$
   (e) $\mathbb{E}[B_s^{100} B_t^{101}]$

4. **Brownian Probabilities** Let B_t be a standard Brownian motion. Write the following probabilities as an integral.
   (a) $P(B_1 > 1, B_2 > 1)$
   (b) $P(B_1 > 1, B_2 > 1, B_3 > 1)$

5. **Time Inversion** Let B_t be a standard Brownian motion and $X_t = tB_{1/t}$ for $t > 0$.
   (a) Show that X_t has the distribution of a Brownian motion on $t > 0$.
   (b) Prove that X_t converges to 0 as $t \to 0$ in the sense of L^2 convergence.
   (c) Prove the following law of large numbers for Brownian motion
$$\lim_{t \to \infty} \frac{X_t}{t} = 0.$$