Lecture #1: Probability Spaces

Example:

Flip a coin, what happens??

- fall on heads, tails
- fall on edge
- roll away
- go through ceiling

⋮

There is a lot of uncertainty, dictated by many things. We build a mathematical universe Ω in which the only outcomes are H, T, i.e.,

$$\Omega = \{H, T\}.$$

We define a probability $P: \Omega \to \mathbb{R}$ by

$$P(\emptyset) = 0,\quad P(H) = \tfrac{1}{2},\quad P(T) = \tfrac{1}{2},\quad P(\Omega) = 1$$

probability nothing happens   probability heads   probability tails   probability something happens.

- Ω is called the sample space
- A subset of Ω is an event.

The possible events are:

$$\{\{H\}, \{T\}, \emptyset, \Omega\}.$$

Definition: A probability P is a function $P: \Omega \to [0,1]$ satisfying

(i) $P(\emptyset) = 0$, $P(\Omega) = 1$

(ii) If $A_1, A_2, \ldots$ is an infinite sequence of mutually exclusive events ($A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Example:

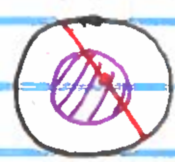
If $\Omega = \{(1,0),(0,1),(1,1),(0,0)\}$. For any $A \in \Omega$, define
$$P(A) = \frac{\#A}{\#\Omega}$$
$$\Rightarrow P((1,0)) = \frac{1}{4},\ P((1,0)\cup(0,1)) = \frac{1}{2}, \ldots$$

Example (Bertrand's Paradox)

Take a circle of radius 2 inches in the plane and choose a chord of this circle at random. What is the probability this chord intersects a concentric circle of radius 1 inch?

Solution 1:

Any chord is determined by the location of its midpoint



$$\Rightarrow \text{probability} = \frac{\text{area of inner circle}}{\text{area of larger circle}} = \frac{1}{4}.$$

Solution 2:

By symmetry under rotation we may assume the chord is vertical

$$\Rightarrow \text{probability} = \frac{\text{diameter of inner circle}}{\text{diameter of outer circle}} = \frac{1}{2}$$

Solution 3:

By symmetry, assume chord is at the far left.

$$\text{Probability} = \frac{2\pi/6}{2\pi/2} = \frac{1}{3}$$

The paradox comes from how we define random.

Proposition-

1. If $A \cap B = \emptyset$ then $P(A \cap B) = P(A) + P(B)$.
2. $P(A^c) = 1 - P(A)$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
4. If $A \subseteq B$ then $P(A) \leq P(B)$.

Theorem- Consider a probability P on Ω. If $A_1, A_2, \ldots$ is an infinite sequence of increasing events, i.e.,

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$$

then

$$P(A_1 \cup A_2 \cup \ldots) = \lim_{n \to \infty} P(A_n).$$

Similarily, if $A_1, A_2, \ldots$ is an infinite set of decreasing events, i.e.,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$$

then

$$P(A_1 \cap A_2 \cap \ldots) = \lim_{n \to \infty} P(A_n).$$

proof:

I will do the case for increasing events. Let

$$A_0 = \emptyset$$
$$B_1 = A_1 \setminus A_0$$
$$B_2 = A_2 \setminus A_1$$
$$\vdots$$
$$B_n = A_n \setminus A_{n-1}$$

Therefore,

$$\begin{aligned} P(A_1 \cup A_2 \cup \ldots) &= P(B_1 \cup B_2 \cup \ldots) \\ &= P(B_1) + P(B_2) + \ldots \\ &= P(A_1) + P(A_2) - P(A_1) + P(A_3) - P(A_2) + \ldots + P(A_n) - P(A_{n-1}) + \ldots \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} (P(A_n) - P(A_{n-1})) = \lim_{N\to\infty} P(A_N). \end{aligned}$$

Definition - A probability space $(\Omega, \mathcal{F}, P)$ is a triple where

- Ω is a set called the sample space
- $\mathcal{F}$ is a collection of subsets called a σ-field and satisfies
  - (i) $\Omega \in \mathcal{F}$
  - (ii) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
  - (iii) If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$
- $P: \Omega \to [0,1]$ satisfying
  - (i) $P(\emptyset) = 0$
  - (ii) $P(\Omega) = 1$
  - (iii) If $A_1, A_2, \ldots$ are disjoint then
    $$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$