Lecture 10: Conditional Expectation and σ-Algebras

Definition: A σ-algebra $\mathcal{F}$ of a sample space Ω is a collection of events $A \in \mathcal{F}$ such that

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Examples:

1. $\{\phi, \Omega\}$ = trivial σ-algebra
2. $\{\phi, \Omega, A, A^c\}$ = σ-algebra generated by an event A.

More Important Example:

Let X_i be a sequence of random variables satisfying

$$P(X_i = 1) = 1/2 \quad P(X_i = -1) = 1/2$$

and define

$$S_n = \sum_{i=1}^{n} X_i.$$

We take $\Omega = \{(x_1, x_2, \ldots) \text{ where } x_i = \pm 1\}$.

1. $S_0 = 0$

$$\Rightarrow P(S_0 \in [a,b]) = P(\{w : 0 \in [a,b]\})$$

$$= \begin{cases} P(\phi) & \text{if } 0 \notin [a,b] \\ P(\Omega) & \text{if } 0 \in [a,b] \end{cases}$$

The only events needed are ϕ, Ω

$$\Rightarrow \mathcal{F}_0 = \underbrace{\{\phi, \Omega\}}_{\text{information you know at } i=0.}$$

2. $S_1 = X_1$

$\Rightarrow P(S_1 \in [a,b]) = P(\{w : X_1 \in [a,b]\})$

$$= \begin{cases} P(\phi) \text{ if } 1 \text{ or } -1 \notin [a,b] \\ P(\Omega) \text{ if } 1 \text{ and } -1 \in [a,b] \\ P(\{w \in \Omega : w = (1,\cdot,\cdot,\ldots)\}) \text{ if } 1 \in [a,b] \text{ and } -1 \notin [a,b] \\ P(\{w \in \Omega : w = (-1,\cdot,\cdot,\ldots)\}) \text{ if } 1 \notin [a,b] \text{ and } 1 \in [a,b] \end{cases}$$

$\Rightarrow \mathcal{F}_2 = \{\phi, \Omega, \{w : w = (1,\cdot,\cdot,\ldots)\}, \{w : w = (-1,\cdot,\cdot,\cdot,\ldots)\}\}$

is a new σ-algebra containing information you know at $i=1$.

$\Rightarrow \mathcal{F}_0 \subset \mathcal{F}_1$ * $\mathcal{F}_1$ is a σ-algebra generated by S_1

3. $S_2 = X_1 + X_2$

Let $A = \{w : w = (1,1,\cdot,\cdot,\ldots)\}$, $B = \{w : w = (-1,-1,\ldots)\}$, $C = \{w : w = (-1,1,\ldots)\}$, $D = \{w : w = (1,-1,\ldots)\}$.

$$\Rightarrow P(S_2 \in I) = \begin{cases} P(\phi) \text{ if } -2,0,2 \notin I \\ P(\Omega) \text{ if } -2,0,2 \in I \\ P(A) \text{ if } 2 \in I, \ -2,0 \notin I \\ P(B) \text{ if } -2 \in I, \ 0,2 \notin I \\ P(C \cup D) \text{ if } 0 \in I, \ -2,2 \notin I \\ P(A \cup B) \text{ if } -2,2 \in I, \ 0 \notin I \\ P(A \cup (C \cup D)) \text{ if } 2,0 \in I, \ -2 \notin I \\ P(B \cup (C \cup D)) \text{ if } -2,0 \in I, \ 2 \notin I \end{cases}$$

$\rightarrow \mathcal{F}_2 = \{\phi, \Omega, \{1,1,\cdots\}, \{-1,-1,\cdots\}, \{1,-1,\cdots\}, \{-1,1,\cdots\}\}$

* $\mathcal{F}_2$ is σ-algebra generated by S_2

$\Rightarrow \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$

Notation: $\sigma(X)$ is the σ-algebra generated by a random variable X. It is the smallest σ-algebra for which X is measurable on $(\Omega, \mathcal{F}, P)$.

Definition- Let X be a random variable on $(\Omega, \mathcal{F}, P)$. Consider another $\mathcal{G} \subset \mathcal{F}$. Then X is said to be $\mathcal{G}$ measurable if and only if

$$\{\omega : X(\omega) \in I\} \in \mathcal{G}$$

for all intervals I in $\mathbb{R}$.

Example:

Let $\mathcal{G} = \{\phi, \Omega\}$. If X is $\mathcal{G}$ measurable then X is a constant.

~~proof~~:

If X is $\mathcal{G}$ measurable then for all $[a,b)$,

$$\{\omega : X(\omega) \in [a,b)\} = \phi \text{ or } \{\omega : X(\omega) \in [a,b)\} = \Omega$$

Now, suppose there exists x_1, x_2 and ω_1, ω_2 such that $X_1(\omega_1) = x_1$ and $X_2(\omega_1) = x_2$. Therefore,

$$\Omega = \bigcap_{n=1}^{\infty} \{\omega : X(\omega) \in [x_1 - 1/n, 1/n)\} = \bigcap_{n=1}^{\infty} \{\omega : X(\omega) \in [x_2 - 1/n, 1/n)\}.$$

$$\Rightarrow \Omega = \{\omega : X(\omega) = x_1\} = \{\omega : X(\omega) = x_2\}.$$

$$\Rightarrow x_1 = x_2.$$

Theorem- If Y is $\sigma(X)$ measurable then $Y = g(X)$ for some function g.

Definition- Let Y be a random variable on $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \leq \mathcal{F}$ be another σ-field. The conditional expectation of Y given $\mathcal{G}$, denoted $\mathbb{E}[Y|\mathcal{G}]$, satisfies

(a) $\mathbb{E}[Y|\mathcal{G}]$ is $\mathcal{G}$ measurable

(b) If W is $\mathcal{G}$ measurable then

$$\mathbb{E}[WY] = \mathbb{E}[W\,\mathbb{E}[Y|\mathcal{G}]].$$

Properties:

1. If Y - $\mathcal{G}$-measurable then
$$\mathbb{E}[Y|\mathcal{G}] = Y$$
2. If Y - $\mathcal{G}$-measurable and X is another random variable then
$$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$$
3. If Y is independent of $\mathcal{G}$, i.e. for all $A \in \mathcal{G}$
$$P(\{Y \in I\} \cap A) = P(\{Y \in A\})P(A)$$
then
$$\mathbb{E}[Y|\mathcal{G}] = \mathbb{E}[Y]$$
4. $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$.
5. $\mathbb{E}[Y^2] = \mathbb{E}[\mathbb{E}[Y|\mathcal{G}]^2] + \mathbb{E}[(Y - \mathbb{E}[Y|\mathcal{G}])^2]$
$$\Rightarrow \mathbb{E}[\mathbb{E}[Y|\mathcal{G}]^2] \leq \mathbb{E}[Y^2].$$

Example:
$$\begin{aligned}
\mathbb{E}[e^{tB_1}|B_{1/2}] &= \mathbb{E}[e^{t(B_1 - B_{1/2} + B_{1/2})}|B_{1/2}] \\
&= \mathbb{E}[e^{tB_{1/2}} e^{t(B_1 - B_{1/2})}|B_{1/2}] \\
&= e^{tB_{1/2}}\mathbb{E}[e^{t(B_1 - B_{1/2})}|B_{1/2}] \\
&= e^{tB_{1/2}}\mathbb{E}[e^{t(B_1 - B_{1/2})}] \\
&= e^{tB_{1/2}} e^{t^2/4} \\
&= e^{tB_{1/2} + t^2/4}
\end{aligned}$$

Recall, for a Gaussian Random variable with mean μ and variance σ^2 that
$$\mathbb{E}[e^{tX}] = e^{\mu t + 2\sigma^2 t^2}.$$