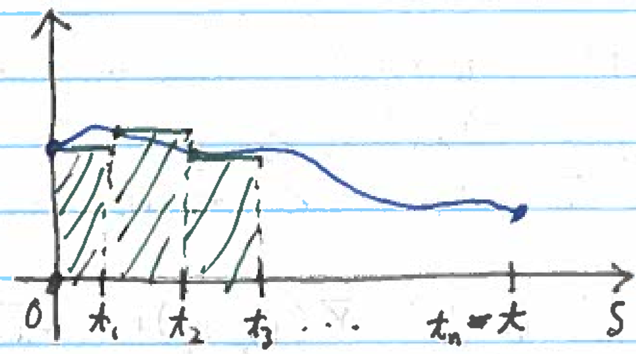
Lecture 13: The Ito Integral

Recall the Riemann integral is defined by

$$G(t) = \int_0^t g(s)\,ds = \lim_{n\to\infty} \sum_{j=0}^{n-1} g(t_j)(t_{j+1} - t_j)$$



Key idea:

Add up sum of functions for which we know the area and take limit.

Definition: A *simple function* on the the interval $[0,t]$ is any function of the form:

$$X(s) = a_0 \mathbb{1}_{[0,t_1)}(t) + a_1 \mathbb{1}_{[t_1,t_2)}(t) + \dots + a_{n-1} \mathbb{1}_{[t_{n-1},t_n]}(t).$$

$$\Rightarrow \int_0^t X(s)\,ds = a_0(t_1 - t_0) + a_1(t_2 - t_1) + \dots + a_{n-1}(t_n - t_{n-1}).$$

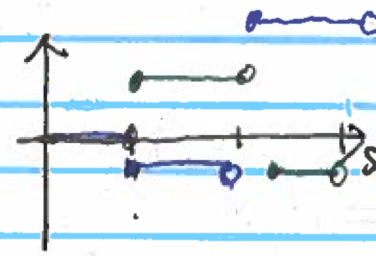
Definition: Let $\mathcal{F}_t$ be a filtration. A *simple random process* on $[0,T]$ is any random variable of the form.

$$X_t = Y_0 \mathbb{1}_{[0,t_1)}(t) + Y_1 \mathbb{1}_{[t_1,t_2)}(t) + \dots + Y_{n-1} \mathbb{1}_{[t_{n-1},t_n]}(t).$$

Where Y_j is $\mathcal{F}_{t_j}$ measurable and $\mathbb{E}[Y_j^2] < \infty$. The space of all simple random variables is denoted $S(T)$.

Example:

$$X_s = \begin{cases} 0, & \text{if } s \in [0, 1/3) \\ B_{1/3}, & \text{if } s \in [1/3, 2/3) \\ B_{2/3}, & \text{if } s \in [2/3, 1) \end{cases}$$



Two different realizations

Definition- Let $X \in S(T)$ be a simple random process given by

$$X = Y_0 \mathbb{1}_{[t_0, t_1)} + \dots + Y_{n-1} \mathbb{1}_{[t_{n-1}, T]}.$$

The ito integral of X with respect to Brownian motion is given by:

$$\int_0^T X_s \, dB_s = Y_0 \cdot (B_{t_1} - B_{t_0}) + Y_1 (B_{t_2} - B_{t_1}) + \dots + Y_{n-1} (B_{t_n} - t_{n-1}).$$

Example:

Taking X_s in the previous example we have:

$$I_t = \int_0^t X_s \, dB_s = \begin{cases} 0 & \text{if } t \in [0, 1/3) \\ B_{1/3}(B_t - B_{1/3}) & \text{if } t \in [1/3, 2/3) \\ B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_t - B_{2/3}) & \text{if } t \in [2/3, 1). \end{cases}$$

I_t is a martingale, For example, take $t > 2/3$ and $1/3 \le s < 2/3$. Then,

$$\mathbb{E}[I_t | \mathcal{F}_s] = \mathbb{E}[B_{1/3}(B_{2/3} - B_{1/3}) + B_{2/3}(B_t - B_{2/3}) | \mathcal{F}_s]$$

$$= B_{1/3} \mathbb{E}[(B_{2/3} - B_{1/3}) | \mathcal{F}_s] + \mathbb{E}[B_{2/3}(B_t - B_{2/3}) | \mathcal{F}_s]$$

$$= B_{1/3} \mathbb{E}[(B_{2/3} - B_s + B_s - B_{1/3} | \mathcal{F}_s] + \mathbb{E}[(B_{2/3} - B_s + B_s)(B_t - B_{2/3}) | \mathcal{F}_s]$$

$$= B_{1/3} \mathbb{E}[(B_{2/3} - B_s) | \mathcal{F}_s] + B_{1/3} \mathbb{E}[B_s - B_{1/3} | \mathcal{F}_s]$$
$$+ \mathbb{E}[(B_{2/3} - B_s)(B_t - B_{2/3}) | \mathcal{F}_s] + \mathbb{E}[B_s (B_t - B_{2/3}) | \mathcal{F}_s]$$

$$= B_{1/3} \mathbb{E}[B_{2/3} - B_s] + B_{1/3}(B_s - B_{1/3}) + (B_{2/3} - B_s) \mathbb{E}[B_t - B_{2/3} | \mathcal{F}_s]$$
$$+ B_s \mathbb{E}[B_t - B_{2/3} | \mathcal{F}_s].$$

$$= B_{1/3}(B_s - B_{1/3}) + (B_{2/3} - B_s) \mathbb{E}[B_t - B_{2/3}] + B_s \mathbb{E}[B_t - B_{2/3}]$$

$$= B_{1/3}(B_s - B_{1/3}) = I_s.$$

Theorem - Let B_t be a standard Brownian motion on $[0,T]$ defined on $(\Omega, \mathcal{F}, P)$ and $X_t, X_t' \in S(T)$. Then.

1. **Linearity** -
$$\int_0^t (aX_s + bX_s')\,dB_s = a\int_0^t X_s\,dB_s + b\int_0^t X_s'\,dB_s.$$
2. **Continuous Martingale** - The process $Y_t = \int_0^t X_s\,dB_s$ is a martingale for the Brownian filtration.
3. **Ito Isometry** - The random variable $Y_t = \int_0^t X_s\,dB_s$ is in $L^2(\Omega, \mathcal{F}, P)$ with mean 0 and variance
$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\left(\int_0^t X_s\,dB_s\right)^2\right]$$
$$= \int_0^t \mathbb{E}[X_s^2]\,ds$$
$$= \mathbb{E}\left[\int_0^t X_s^2\,ds\right]$$

~~proof~~:

1. Trivial
2. Let $X_s \in S(T)$. Then,
$$X_t = \sum_{i=0}^{k-1} Z_i \mathbb{1}_{[t_i, t_{i+1}]}(t)$$
$$\Rightarrow Y_t = \int_0^t X_s\,dB_s$$
$$= \sum_{i=0}^{k-1} X_i (B_{t_{i+1}-t_i})$$

Now,
$$\mathbb{E}[X_i(B_{i+1} - B_i) \mid \sigma(s)] = \begin{cases} X_i(B_{t_{i+1}} - B_{t_i}), & \text{if } s > i+1 \\ X_i\,\mathbb{E}[B_{t_{i+1}} \mid \sigma(B_s)] - X_i B_{t_i}, & \text{if } i < s < i+1 \\ \mathbb{E}[X_i(B_{t_{i+1}} - B_{t_i}) \mid \sigma(B_s)], & \text{if } s < i. \end{cases}$$
$$= \begin{cases} X_i(B_{t_{i+1}} - B_{t_i}), & \text{if } s > i+1 \\ X_i(B_s - B_{t_i}), & \text{if } i < s < i+1 \\ \mathbb{E}[X_i \underbrace{\mathbb{E}[B_{t_{i+1}} - B_{t_i} \mid \sigma(B_{t_i})]}_{=0} \mid \sigma(B_s)] & \text{if } s < i \end{cases}$$
$$\Rightarrow \mathbb{E}[Y_t \mid \sigma(B_s)] = X_0(B_{t_1} - B_0) + X_1(B_{t_2} - B_{t_1}) + \dots + X_{i^*}(B_s - B_{t_{i^*}})$$
$$= Y_s.$$

3. Since $\mathbb{E}[Z_j(B_{t_{j+1}}-B_{t_j})] = \mathbb{E}[\mathbb{E}[Z_j(B_{t_{j+1}}-B_{t_j}) \mid \sigma(B_{t_j})]]$
$= \mathbb{E}[Z_j\,\mathbb{E}[B_{t_{j+1}}-B_{t_j} \mid \sigma(B_{t_j})]]$
$= \mathbb{E}[Z_j \cdot 0]$
$= 0.$

Therefore, $\mathbb{E}[Y_t] = \mathbb{E}[\sum_{j=0}^{n-1} Z_j(B_{t_{j+1}}-B_{t_j})] = 0.$

Now,
$$\mathbb{E}[Y_t^2] = \mathbb{E}[(\int_0^t X_s\,dB_s)^2]$$
$$= \mathbb{E}[(\sum_{j=0}^{n} Z_j(B_{t_{j+1}}-B_{t_j}))^2]$$
$$= \mathbb{E}[\sum_{i,j}^{n} Z_j Z_i (B_{t_{j+1}}-B_{t_j})(B_{t_{i+1}}-B_{t_i})]$$

If $t_i \neq t_j$ and without loss of generality $t_i < t_j$ then
$$\mathbb{E}[Z_j Z_i (B_{t_{j+1}}-B_{t_j})(B_{t_{i+1}}-B_{t_i})] = \mathbb{E}[Z_j Z_i (B_{t_{i+1}}-B_{t_i})\,\mathbb{E}[B_{t_{j+1}}-B_{t_j} \mid \sigma(B_{t_j})]]$$
$$= \mathbb{E}[Z_j Z_i (B_{t_{i+1}}-B_{t_i}) \cdot 0]$$
$$= 0.$$

If $t_i = t_j$ then
$$\mathbb{E}[Z_j Z_i (B_{t_{j+1}}-B_{t_j})(B_{t_{i+1}}-B_{t_i})] = \mathbb{E}[Z_i^2 (B_{t_{i+1}}-B_{t_i})^2]$$
$$= \mathbb{E}[Z_i^2\,\mathbb{E}[(B_{t_{i+1}}-B_{t_i})^2 \mid \sigma(B_{t_i})]]$$
$$= \mathbb{E}[Z_i^2 (t_{i+1}-t_i)]$$
$$= (t_{i+1}-t_i)\,\mathbb{E}[Z_i^2]$$

Therefore,
$$\mathbb{E}[Y_t^2] = \sum_{j=0}^{n-1} \mathbb{E}[Y_j^2](t_{j+1}-t_j) = \int_0^t \mathbb{E}[X_s^2]\,ds.$$

<u>Definition</u> – The <u>Ito integral</u> for any "integrable" stochastic process is the limit
$$\int_0^t X_s\,dB_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} X_i(B_{i+1}-B_i).$$
All of the previous three properties hold for this limit.

Corollary: Let B_t be standard Brownian motion. Then,

1. $\mathbb{E}[(\int_0^t X_s\,dB_s)(\int_0^{t'} X_s\,dB_s)] = \int_0^{\min\{t,t'\}} \mathbb{E}[X_s^2]\,ds$
2. $\mathbb{E}[(\int_0^t X_s\,dB_s)(\int_0^t Y_s\,dB_s)] = \int_0^t \mathbb{E}[X_s Y_s]\,ds.$

~~proof~~

2. $$\mathbb{E}[(\int_0^t (X_s+Y_s)\,dB_s)^2] = \int_0^t \mathbb{E}[(X_s+Y_s)^2]\,ds$$
$$= \int_0^t (\mathbb{E}[X_s^2] + 2\mathbb{E}[X_s Y_s] + \mathbb{E}[Y_s^2])\,ds$$

However,

$$\mathbb{E}[(\int_0^t (X_s+Y_s)\,dB_s)^2] = \mathbb{E}[(\int_0^t X_s\,dB_s + \int_0^t Y_s\,dB_s)^2]$$
$$= \mathbb{E}[(\int_0^t X_s\,dB_s)^2] + 2\mathbb{E}[(\int_0^t X_s\,dB_s)(\int_0^t Y_s\,dB_s)] + \mathbb{E}[(\int_0^t Y_s\,dB_s)^2]$$
$$= \int_0^t \mathbb{E}[X_s^2]\,ds + 2\mathbb{E}[(\int_0^t X_s\,dB_s)(\int_0^t Y_s\,dB_s)] + \int_0^t \mathbb{E}[Y_s^2]\,ds$$

$$\Rightarrow \mathbb{E}[(\int_0^t X_s\,dB_s)(\int_0^t Y_s\,dB_s)] = \int_0^t \mathbb{E}[X_s Y_s]\,ds.$$

Example:

$I_t = \int_0^t B_s\,dB_s$, $J_t = \int_0^t B_s^2\,dB_s$.

$\mathbb{E}[I_t] = 0$, $\mathbb{E}[J_t] = 0$

$$\mathbb{E}[I_t^2] = \mathbb{E}[(\int_0^t B_s\,dB_s)^2] = \int_0^t \mathbb{E}[B_s^2]\,ds = \int_0^t s\,ds = t^2/2$$
$$\mathbb{E}[J_t^2] = \mathbb{E}[\int_0^t B_s^2\,dB_s] = \int_0^t \mathbb{E}[B_s^4]\,ds = \int_0^t 3s^2\,ds = t^2$$

$$\mathbb{E}[I_t J_t] = \mathbb{E}[(\int_0^t B_s\,dB_s)(\int_0^t B_s^2\,dB_s)] = \mathbb{E}[\int_0^t \mathbb{E}[B_s^3]\,ds] = 0.$$

Example:

$X_t = \int_0^t B_s \, dB_s$ and let $Y_t = \int_0^s X_s \, dB_s$.

$\Rightarrow \mathbb{E}[X_t] = 0$

$$\Rightarrow \mathbb{E}[Y_t^2] = \int_0^t \mathbb{E}[X_s^2] ds$$
$$= \int_0^t \int_0^s \mathbb{E}[B_u^2] du\, ds$$
$$= \int_0^t \int_0^s u \, du\, ds$$
$$= t^3/6.$$

Corollary: Let B_t be a standard Brownian motion and f be be a function such that $\int_0^T f^2(x) dx < \infty$. Then

$$X_t = \int_0^t f(s) dB_s$$

is a Gaussian process with mean 0 and covariance.

$$Cov(X_t, X_{t'}) = \int_0^{\min\{t, t'\}} f(s)^2 ds.$$

proof:

The approximation on a simple function is

$$\int_0^t f(s) dB_s = \sum_{j=0}^{n-1} f(t_j)(B_{t_{j+1}} - B_{t_j}).$$

Sum of Gaussians $\Rightarrow$ Gaussian.