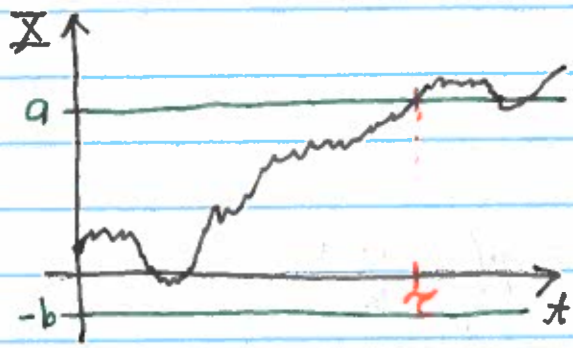
Lecture 16: Hitting Times

Example:

$d\mathbb{X} = \nu\,dt + \sigma\,dB$ ← Brownian motion with drift

$\mathbb{X}(0) = x_0$



$\tau = \min\{t \geq 0 : \mathbb{X}_\tau > a \text{ or } \mathbb{X}_\tau < -b\}$

What is $P(\mathbb{X}_\tau = a)$ and what is $\mathbb{E}[\tau]$?

Idea: Construct a martingale $M_t = g(\mathbb{X})$ such that

$g(a) = 1$ and $g(-b) = 0$

$\Rightarrow \mathbb{E}[M_\tau] = \mathbb{E}[M_0] = \mathbb{E}[g(\mathbb{X}(0))] = g(x_0)$

Therefore,

$$g(x_0) = \mathbb{E}[M_\tau] = P(\mathbb{X}_\tau = a)g(a) + P(\mathbb{X}_\tau = -b)g(b)$$

$$\Rightarrow g(x_0) = P(\mathbb{X}_\tau = a)$$

Now, by Ito's formula

$$g(\mathbb{X}) - g(\mathbb{X}_0) = \int_0^t \frac{dg}{dx}\,dx + \frac{1}{2}\int_0^t \frac{d^2g}{dx^2}\,dx^2$$

$$= \int_0^t \frac{dg}{dx}\left(\nu\,dt + \sigma\,dB\right) + \frac{1}{2}\int_0^t \frac{d^2g}{dx^2}\sigma^2\,dt$$

Therefore, in for g to be a martingale we need

$$\nu \frac{dg}{dx} + \frac{1}{2}\sigma^2 \frac{d^2 g}{dx^2} = 0 \longrightarrow$$ linear with constant coefficients.

$$g(-b) = 0, \quad g(a) = 1$$

We can integrate once

$$\nu g(x) + \frac{1}{2}\sigma^2 \frac{dg}{dx} = c_1$$

$$\Rightarrow \frac{dg}{dx} + \frac{2\nu}{\sigma^2} g(x) = c_1$$

$$\Rightarrow \frac{d}{dx}\left(e^{2\nu/\sigma^2 x} g(x)\right) = c_1 \exp\left(\frac{2\nu x}{\sigma^2}\right)$$

$$\Rightarrow \int_{-b}^{x} d\left(\exp\left(\frac{2\nu}{\sigma^2} y\right) g(y)\right) = \int_{-b}^{x} c_1 \exp\left(\frac{2\nu y}{\sigma^2}\right) dy$$

$$\Rightarrow \exp\left(\frac{2\nu}{\sigma^2} x\right) g(x) = c_1\left(\exp\left(\frac{-2\nu b}{\sigma^2}\right) - \exp\left(\frac{2\nu x}{\sigma^2}\right)\right)$$

$$\Rightarrow \exp\left(\frac{2\nu a}{\sigma^2}\right) = c_1\left(\exp\left(\frac{-2\nu b}{\sigma^2}\right) - \exp\left(\frac{2\nu a}{\sigma^2}\right)\right)$$

$$\Rightarrow c_1 = \frac{1}{e^{-2\nu/\sigma^2 (b+a)} - 1}$$

Therefore,

$$g(x_0) = P(X_\tau = a) = \frac{1}{e^{-2\nu/\sigma^2 (b+a)} - 1} \cdot e^{-2\nu/\sigma^2 (b+x_0)} - 1$$

$$\Rightarrow \boxed{P(X_\tau = a) = \frac{1 - e^{-2\nu/\sigma^2 (b+x_0)}}{1 - e^{-2\nu/\sigma^2 (a+b)}}}$$

To find $\mathbb{E}[X_\tau]$ we seek a martingale in the form

$$N_t = t + f(X_t)$$

$$\Rightarrow \mathbb{E}[N_\tau] = \mathbb{E}[\tau] + \mathbb{E}[f(X_\tau)]$$

$$= \mathbb{E}[\tau] + f(a)P(X_\tau = a) + f(-b)P(X_\tau = -b)$$

$$\Rightarrow \mathbb{E}[N_\tau] = \mathbb{E}[N_0]$$

$$= 0 + f(X_0)$$

Therefore, if $f(a) = f(-b) = 0$ we have

$$f(X_0) = \mathbb{E}[\tau].$$

By Ito's Formula

$$d(t + f(x)) = dt + \frac{df}{dx}dx + \frac{1}{2}\frac{d^2f}{dx^2}dx^2$$

$$= dt + \frac{df}{dx}(\nu dt + \sigma dB) + \frac{1}{2}\frac{d^2f}{dx^2}\sigma^2 dt$$

Therefore, if N is a martingale we need

$$\frac{d^2f}{dx^2} + \frac{2\nu}{\sigma^2}\frac{df}{dx} = -\frac{2}{\sigma^2}$$

$$f(-b) = f(a) = 0.$$

$$\Rightarrow \frac{df}{dx} + \frac{2\nu}{\sigma^2}f = \frac{-2}{\sigma^2}x + C$$

$$\Rightarrow \frac{d}{dx}\left(e^{2\nu/\sigma^2 x} f\right) = \frac{-2}{\sigma^2}xe^{2\nu/\sigma^2 x} + Ce^{2\nu/\sigma^2 x}$$

$$\Rightarrow e^{2\nu/\sigma^2}f(x) = -\frac{2}{\sigma^2}\int_{-b}^{x}\left(ye^{2\nu/\sigma^2 y} + ce^{2\nu/\sigma^2 y}\right)dy$$

Example:

$$dS = \mu S\,dt + \sigma S\,dB$$
$$S(0) = 1$$
$$\tau = \min\{t : S \geq 2 \text{ or } S \leq 1/2\}$$

What is $P(S_\tau = 2)$? What is $\mathbb{E}[\tau]$?

Let $M_t = g(S_t)$ be a martingale satisfying $g(1/2) = 0$ and $g(2) = 1$.
Therefore,

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = g(S_0) = g(1)$$

Furthermore,

$$g(1) = P(S_\tau = 2)\,g(2) + P(S_\tau = 1/2)\,g(1/2)$$
$$\Rightarrow P(S_\tau = 2) = g(1).$$

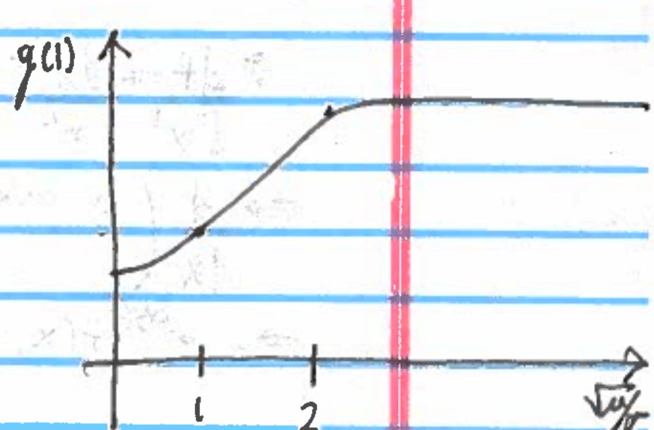
By Ito's formula

$$\frac{dg}{\partial s}\,ds + \frac{1}{2}\frac{\partial^2 g}{\partial s^2}\,ds^2 = dg$$

$$\Rightarrow \frac{dg}{ds}(\mu S\,dt + \sigma S\,dB) + \frac{1}{2}\frac{d^2 g}{ds^2}\sigma^2 S^2\,dt = dg.$$

Therefore, for g to be a martingale we need

$$\boxed{\begin{gathered}\frac{1}{2}\sigma^2 S^2 \frac{d^2 g}{ds^2} + \mu S \frac{dg}{ds} = 0 \\ g(1/2) = 0, \quad g(2) = 1\end{gathered}}$$



$$\boxed{\begin{gathered}\rightarrow g(S) = \frac{\operatorname{erf}(\sqrt{\mu}/2\sigma) - \operatorname{erf}(S\sqrt{\mu}/\sigma)}{\operatorname{erf}(\sqrt{\mu}/2\sigma) - \operatorname{erf}(2\sqrt{\mu}/\sigma)} \\ \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt\end{gathered}}$$

Example:

$$dS = \nu S\,dt + \sigma S\,dB$$

$$S(0) = 1$$

$$\tau = \min\{t : S \geq 2 \text{ or } S \leq 1/2\}.$$

What is $P(S_\tau = 2)$? What is $\mathbb{E}[\tau]$?

Let $M_t = g(S_t)$ be a martingale satisfying $g(1/2) = 0$ and $g(2) = 1$. Therefore,

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = g(S_0) = g(1)$$

Furthermore,

$$g(1) = \mathbb{E}[M_\tau] = P(S_\tau = 2)g(2) + P(S_\tau = 1/2)g(1/2)$$

$$\Rightarrow g(1) = P(S_\tau = 2).$$

By Ito's formula

$$\frac{dg}{dS}dS + \frac{1}{2}\frac{d^2g}{dS^2}dS^2 = dg$$

$$\Rightarrow \frac{dg}{dS}\left(\nu S\,dt + \sigma S\,dB\right) + \frac{1}{2}\frac{d^2g}{dS^2}\sigma^2 S^2\,dt = dg$$

Therefore, for g to be a martingale we need

$$\boxed{\begin{gathered}\frac{1}{2}\sigma^2 S^2 \frac{d^2g}{dS^2} + \nu S\frac{dg}{dS} = 0 \\ g(1/2) = 0,\ g(2) = 1.\end{gathered}}$$

$$\Rightarrow g(S) = \frac{2^{1+2\nu/\sigma^2} S^{1-2\nu/\sigma^2}}{4 - 16^{\nu/\sigma^2}} - \frac{4^{2\nu/\sigma^2}}{4 - 16^{\nu/\sigma^2}}$$

$$\Rightarrow \boxed{g(1) = P(S_\tau = 2) = \frac{2^{1+2\nu/\sigma^2} - 4^{2\nu/\sigma^2}}{4 - 16^{\nu/\sigma^2}}}$$

To find $\mathbb{E}[\tau]$ we need a martingale of the form

$$N_t = t + f(S)$$

$$\Rightarrow \mathbb{E}[N_\tau] = \mathbb{E}[N_0] = \mathbb{E}[\tau] + \mathbb{E}[f(S_\tau)]$$

$$\Rightarrow f(S_0) = \mathbb{E}[\tau] + P(S_\tau = 2)f(2) + P(S_\tau = 1/2)f(1/2).$$

Therefore, if $f(2) = f(1/2) = 0$ we have

$$\mathbb{E}[\tau] = f(S_0) = f(1).$$

Now, by Ito's formula we have

$$dN = dt + \frac{df}{\partial S} dS + \frac{1}{2}\frac{d^2 f}{\partial S^2} dS^2$$

$$= dt + \frac{df}{dS}(\nu S dt + \sigma S dB) + \frac{1}{2}\frac{d^2 f}{dS^2}(\sigma^2 dt).$$

Thus f satisfies the following O.D.E.

$$\boxed{\begin{array}{l} \dfrac{\sigma^3}{2}\dfrac{d^2 f}{dt^2} + \nu \dfrac{df}{dS} = -1 \\ f(1/2) = f(2) = 0 \end{array}}$$

The full solution is too long to write, but

$$\boxed{\mathbb{E}[\tau] = f(1) = \frac{2\left(16^{\nu/\sigma^2}\ln(2) - 2^{1+2\nu}\ln(4) + \ln(16)\right)}{(4-16^{\nu})(2\nu-1)}}$$