Lecture #2: Random Variables

Definition - A random variable is a function X from Ω to $\mathbb{R}$, i.e., $X: \Omega \to \mathbb{R}$.

Example:

Flip a coin twice. $X(\omega)$ = sum of heads that appear.

Notation:

$$\left.\begin{aligned}
P(X=2) &= P(\{\omega : X(\omega)=2\}) \\
&= P(X^{-1}(2)) \\
&= 1/4 \\
P(X=1) &= P(\{\omega : X(\omega)=1\}) \\
&= P(X^{-1}(1)) \\
&= P((0,1)\cup(1,0)) \\
&= 1/2 \\
P(X=0) &= P(\{\omega : X(\omega)=0\}) \\
&= P((0,0)) \\
&= 1/4
\end{aligned}\right\}$$

defines new probability on $\Omega' = \{0,1,2\}$!

$$\begin{aligned}
P(X=-1) &= P(\omega : X(\omega)=-1) \\
&= P(\emptyset) \\
&= 0 \\
P(X \geq 1) &= P(\{\omega : X(\omega) \geq 1\}) \\
&= 3/4
\end{aligned}$$

Example:

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

is a random variable.

Definition - The distribution of a random variable X, denoted ξ_X is given by

$$\xi_X((a,b]) = P(X \in (a,b])$$
$$= P(\{\omega \in \Omega : X(\omega) \in (a,b]\}).$$

The cummulative distribution is a function $F_X : \mathbb{R} \to [0,1]$ defined by
abbreviated CDF

$$F_X(x) = P(X \le x)$$
$$\Rightarrow \xi_X((a,b]) = F_X(b) - F_X(a)$$

If F_X is differentiable, there exists a probability density function $f(x)$ such that

$$\xi_X((a,b]) = F_X(b) - F_X(a)$$
$$= \int_a^b \frac{dF}{dx}\,dx$$
$$= \int_a^b f(x)\,dx$$

*The key point is that a distribution defines a probability on $\mathbb{R}$!

Example:

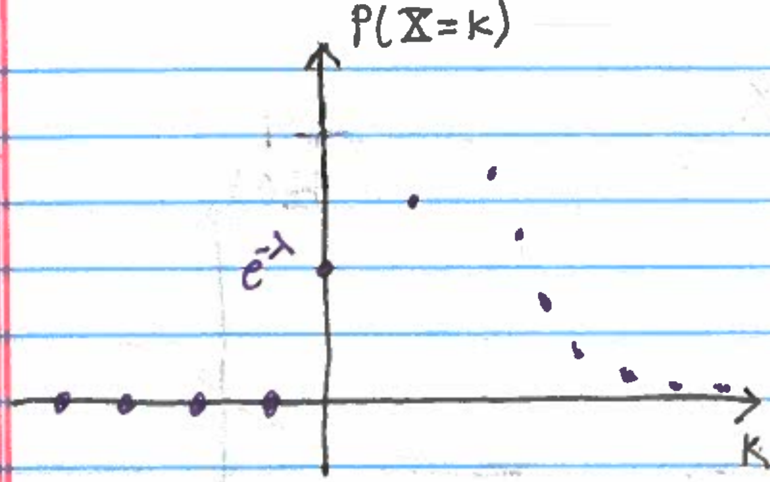
1. Poisson Distribution -

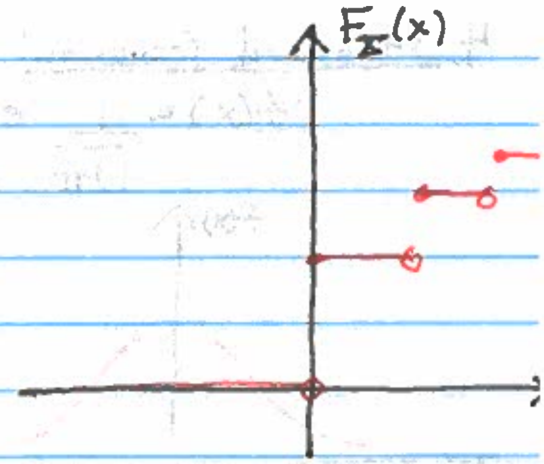
$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k \in \{0,1,2,\dots\}$$

$$\sum_{i=0}^{\infty} \lambda^i/i!\, e^{-\lambda} = e^{-\lambda} \sum_{i=0}^{\infty} \lambda^i/i! = e^{-\lambda} e^{\lambda} = 1.$$

The CDF is

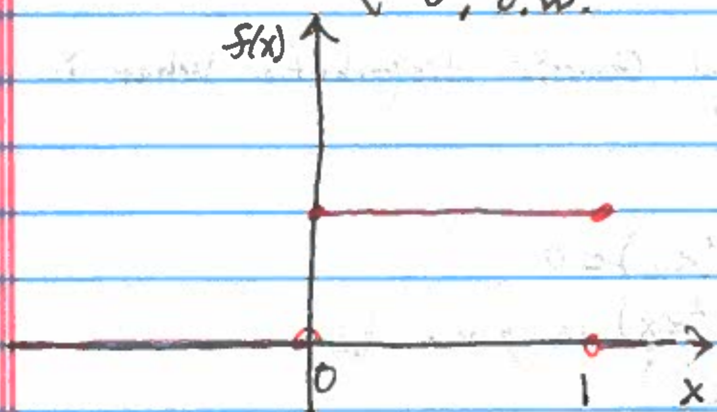
$$F_X(x) = \begin{cases} 0 & x < 0 \\ \sum_{j=0}^{k} \lambda^j/j!\, e^{-\lambda} & \text{if } k \le x < k \end{cases}$$

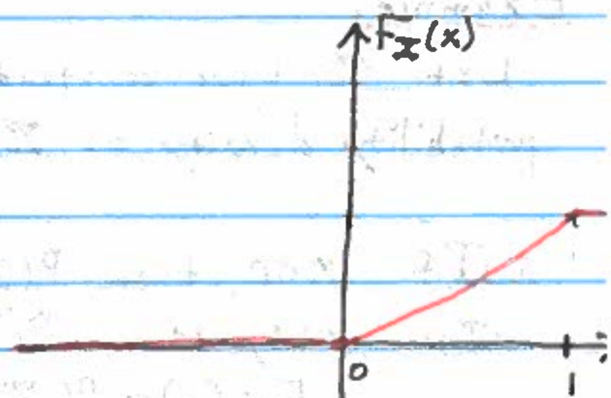




2. Uniform Distribution -

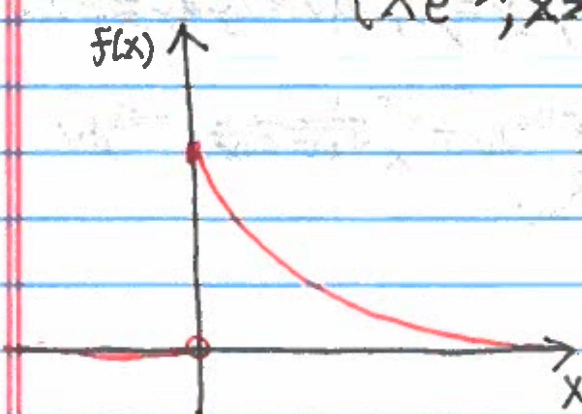
$$f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & o.w. \end{cases}$$

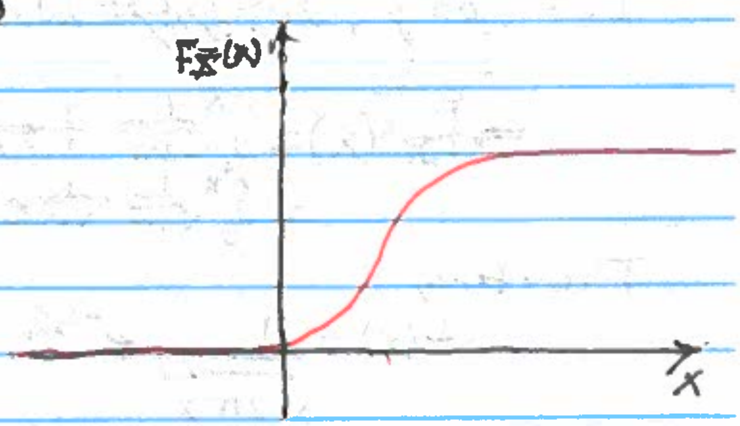




3. Exponential Distribution -

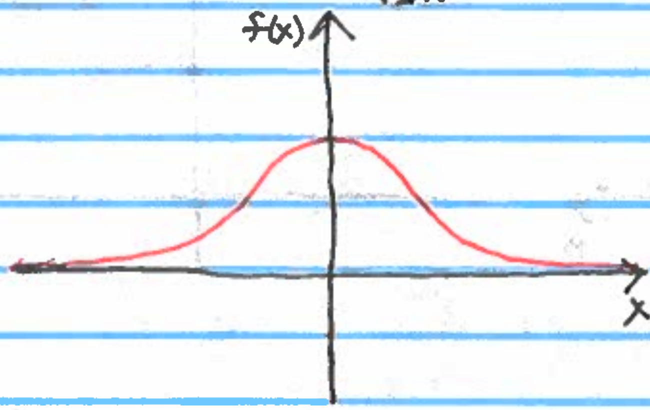
$$f(x) = \begin{cases} 0, & x \leq 0 \\ \lambda e^{-\lambda x}, & x > 0 \end{cases}$$

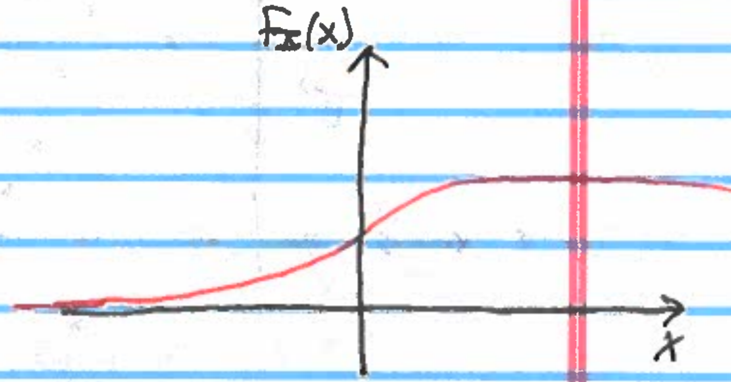




4. Standard Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$





Example:

Let X have a standard Gaussian distribution. What is the probability density of X^2?

- If $x<0$, then $P(X^2<x)=0$
- If $x>0$, then $P(X^2<x)$ is given by

$$F_{X^2}(x) = P(X^2<x)$$
$$= P(0<X^2<x)$$
$$= P(0<X<\sqrt{x} \text{ or } -\sqrt{x}<X<0)$$
$$= \int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \int_{-\sqrt{x}}^{0} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Let $u=y^2$, $du = 2y\,dy = 2\sqrt{u}\,dy$, $v=y^2$, $dv = 2y\,dy = -2\sqrt{v}\,dv$

$$\Rightarrow F_{X^2}(x) = \int_0^x \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{u}} e^{-u/2} du - \int_x^0 \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{v}} e^{-v/2} dv$$
$$= \int_0^x \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{u}} e^{-u/2} du.$$

Therefore,

$$f(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}.$$