Homework #6
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

Problem 1. For each of the following systems, find the fixed fixed points, sketch the nullclines, and then plot a plausible phase portrait. You do not need to analyze the stability of any fixed points.

(a) $\dot{x} = x - y$ and $\dot{y} = 1 - e^x$.

(b) $\dot{x} = x(x - y)$ and $\dot{y} = y(2x - y)$.

(c) $\dot{x} = x^2 - y$ and $\dot{y} = x - y$.

(d) $\dot{x} = 2xy$ and $\dot{y} = y^2 - x^2$.

Problem 2. For each of the following systems, find the fixed points, classify them, sketch the nullclines, and then plot a plausible phase portrait.

1. $\dot{x} = x - y$ and $\dot{y} = x^2 - 4$.
2. $\dot{x} = y + x - x^3$ and $\dot{y} = -y$.
3. $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.
4. $\dot{x} = xy - 1$ and $\dot{y} = \cos(x)$.

Problem 3. A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which

(a) there is a single trajectory that connects the saddles,

(b) there is no trajectory that connects the saddles.

Problem 4. Sketch a phase portrait that has exactly three closed orbits and one fixed point. Be sure to include plausible trajectories inside and outside all the closed orbits.

Problem 5. Consider the system $\dot{x} = -y - x^3$ and $\dot{y} = x$.

(a) Show that the local linearization predicts that the origin is a center.

(b) By converting the system to polar coordinates, show that the origin is in fact a stable spiral.

Problem 6. Classify the fixed point at the origin for $\dot{x} = -y + ax^3$ and $\dot{y} = x + ay^3$ for all real values of the parameter a.

Problem 7. Consider the following model for the interaction of the population of deer and rabbits

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{\kappa_1}\right) - \alpha N_1 N_2,$$
$$\dot{N}_2 = r_2 N_2 \left(1 - \frac{N_1}{\kappa_2}\right) - \beta N_1 N_2,$$

where $r_1, r_2, \kappa_1, \kappa_2, \alpha$, and β are constants.

(a) Give biological interpretations of each of the parameters.

(b) Nondimensionalize this system. There are many ways to do this. You should do the most natural one that makes sense biologically and makes the problem as simple as possible.

(c) Classify the fixed points for this system and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

Problem 8. Consider the system $\ddot{x} = x^3 - x$.

(a) Write this second order differential equation as a system of first order differential equations.

(b) Find a conserved quantity for this system.

(c) Find all the equilibrium and classify them.

(d) Sketch the phase portrait.

(e) Find an equation for any heteroclinic or homoclinic orbits.

Problem 9. Consider the system $\ddot{x} = x - x^2$.

(a) Write this second order differential equation as a system of first order differential equations.

(b) Find a conserved quantity for this system.

(c) Find all the equilibrium and classify them.

(d) Sketch the phase portrait.

(e) Find an equation for any heteroclinic or homoclinic orbits.

Problem 10. In a modified version of the SIS model in which those that recover from the disease are no longer susceptible to the disease we obtain the following model:

$$\dot{S} = -\beta SI \text{ and } \dot{I} = \beta SI - \alpha I,$$

where $\alpha, \beta > 0$.

(a) Find and classify the fixed points, sketch the nullclines, and sketch all possible phase portraits. What happens as $t \to \infty$?

(b) Find a conserved quantity for this system. **Hint:** Find a differential equation for $\frac{dI}{dS}$, separate the variables and integrate both sides.

(c) An epidemic is said to occur if $I(t)$ increases initially. Under what conditions does this occur?

Problems to be completed by graduate students

Problem 11. Consider the following idealized model for three competing species

$$\dot{P} = P(R - S),$$
$$\dot{R} = R(S - P),$$
$$\dot{S} = S(P - R),$$

where P, R, and S are all positive and denote the sizes of the three species P, R, and S.

(a) This system is commonly called a "rock-paper-scissors" system. Explain the various terms in these equations and how it relates to game "rock-paper-scissors". State some of the biological assumptions that being made her implicitly.

(b) Show that $P + R + S$ is a conserved quantity and explain what this means biologically.

(c) Show that PRS is also a conserved quantity.

(d) How does this system behave as $t \to \infty$? Prove that your answer is correct.

Homework #6

#1

(a) $\dot{x} = x - y$

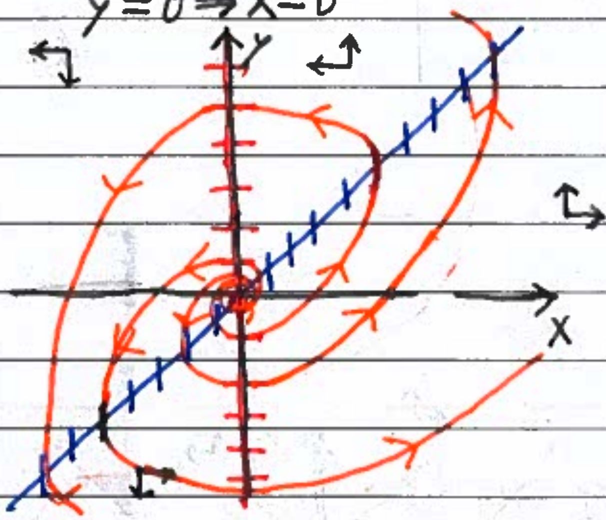
$\dot{y} = 1 - e^x$

Solution:

Nullclines:

$\dot{x} = 0 \Rightarrow y = x$

$\dot{y} = 0 \Rightarrow x = 0$



(b) $\dot{x} = x(x-y)$

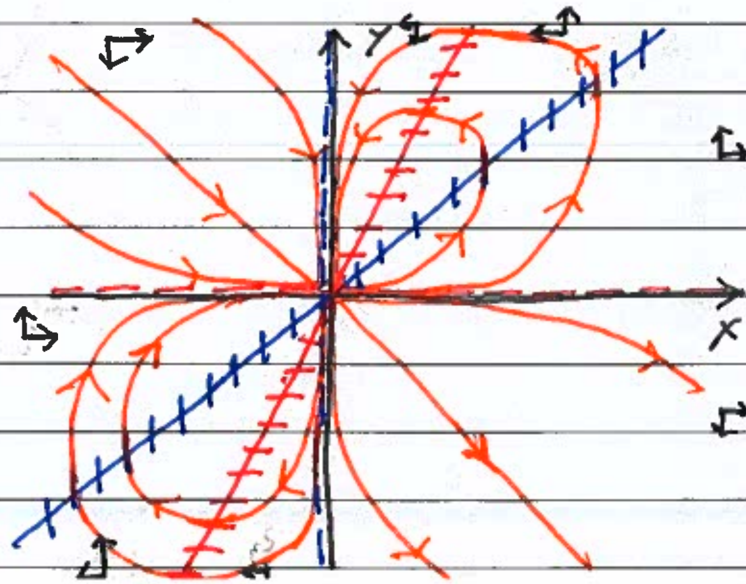
$\dot{y} = y(2x-y)$

Solution:

Nullclines:

$\dot{x} = 0 \Rightarrow x = 0,\ y = x$

$\dot{y} = 0 \Rightarrow y = 0,\ y = 2x$



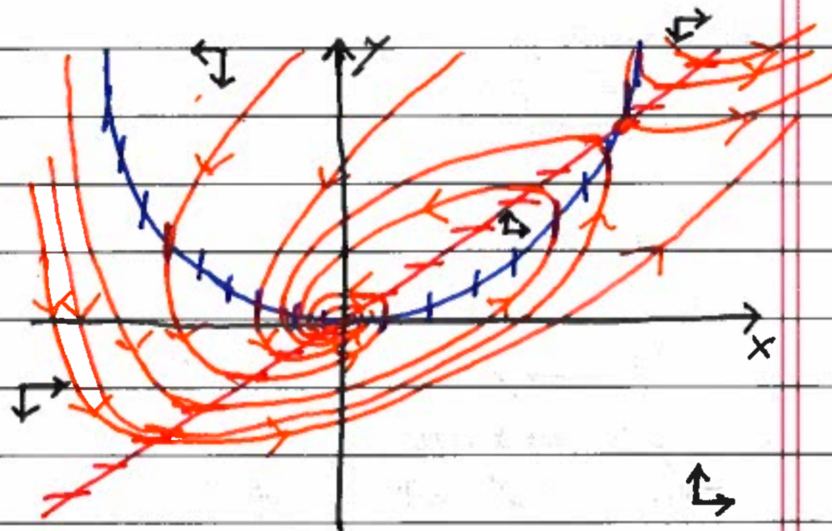
(c) $\dot{x} = x^2 - y$
$\dot{y} = x - y$

Solution:

Nullclines:

$\dot{x} = 0 \Rightarrow y = x^2$

$\dot{y} = 0 \Rightarrow y = x$



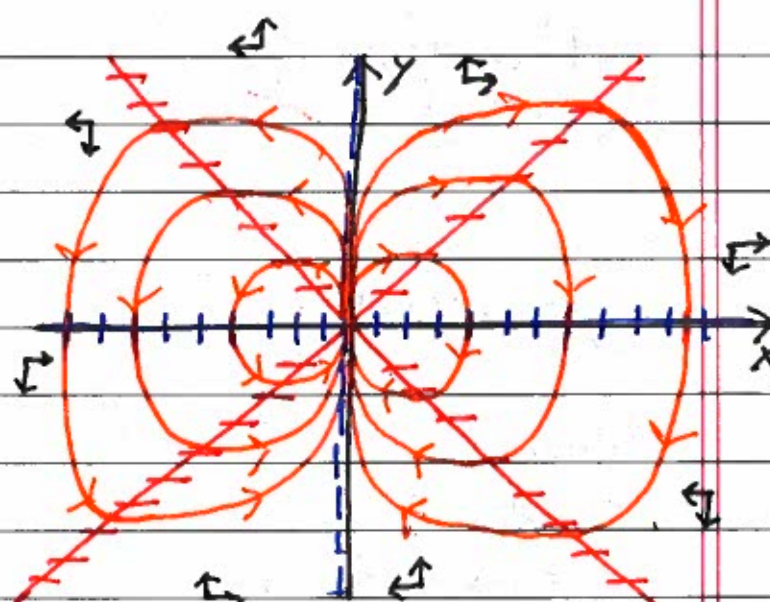
(d) $\dot{x} = 2xy$
$\dot{y} = y^2 - x^2$

Solution:

Nullclines:

$\dot{x} = 0 \Rightarrow x = 0$ or $y = 0$

$\dot{y} = 0 \Rightarrow y = \pm x$



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(c) $\dot{x} = xy - 1$
$\dot{y} = x - y^3$

Solution:

Nullclines:

$\dot{x} = 0$: $y = 1/x$

$\dot{y} = 0$: $x = y^{1/3}$

Therefore, the fixed points are $(-1,-1)$ and $(1,1)$. The Jacobian is

$$J = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}$$

$J(-1,-1)$:

$$\begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$$

$\Rightarrow \lambda_1 + \lambda_2 = -4$

$\lambda_1 \lambda_2 = 4$

$\Rightarrow \lambda_1(\lambda_1 + 4) = -4$

$\Rightarrow \lambda_1^2 + 4\lambda_1 + 4 = 0$

$\Rightarrow (\lambda_1 + 2)^2 = 0$

$\Rightarrow (-1,-1)$ is stable

$J(1,1)$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$\Rightarrow \lambda_1 + \lambda_2 = -2$

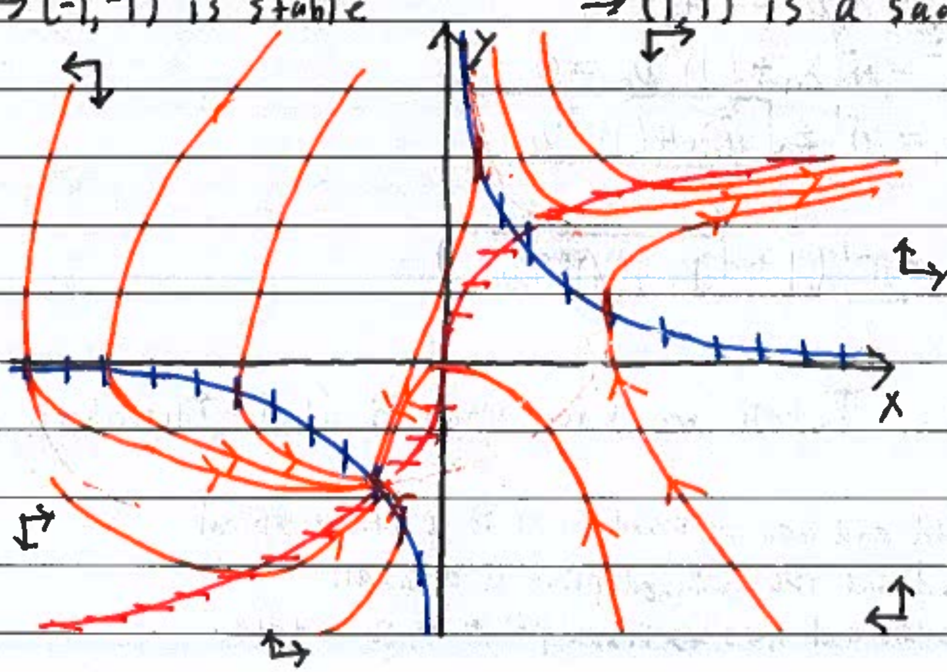
$\Rightarrow \lambda_1 \lambda_2 = -4$

$\Rightarrow \lambda_1(2 + \lambda_1) = 4$

$\Rightarrow \lambda_1^2 + 2\lambda_1 - 4 = 0$

$\Rightarrow \lambda_1 = (-2 \pm \sqrt{-4+16})/2$

$\Rightarrow (1,1)$ is a saddle.



(d) $\dot{x} = xy - 1$
$\dot{y} = \cos(x)$

Solution:

Nullclines:

$\dot{x} = 0$: $y = 1/x$

$\dot{y} = 0$: $x = \pi/2 + n\pi$, $n \in \mathbb{Z}$.

Therefore, the fixed points are $(1/(\pi/2 + n\pi), \pi/2 + n\pi)$ where $n \in \mathbb{Z}$. The Jacobian matrix is

$$J = \begin{bmatrix} y & x \\ -\sin(x) & 0 \end{bmatrix}$$

$J((\pi/2 + n\pi)^{-1}, \pi/2 + n\pi)$:

$$\begin{bmatrix} (\pi/2 + n\pi)^{-1} & (\pi/2 + n\pi) \\ (-1)^{n+1} & 0 \end{bmatrix} = \begin{bmatrix} \nu_n^{-1} & \nu_n \\ (-1)^{n+1} & 0 \end{bmatrix},$$

where $\nu_n = (\pi/2 + n\pi)$. Therefore,

$$\lambda_1 + \lambda_2 = \nu_n^{-1}$$
$$\lambda_1 \lambda_2 = (-1)^n \nu_n$$
$$\Rightarrow \lambda_1(\nu_n^{-1} - \lambda_1) = (-1)^n \nu_n$$
$$\Rightarrow \lambda_1^2 - \lambda_1 \nu_n^{-1} = (-1)^{n+1} \nu_n$$
$$\Rightarrow \lambda_1^2 - \nu_n^{-1} \lambda_1 + (-1)^n \nu_n = 0$$
$$\Rightarrow \lambda_1 = \frac{\nu_n^{-1} \pm \sqrt{\nu_n^{-2} + 4(-1)^{n+1} \nu_n}}{2}$$
$$= \frac{\nu_n^{-1}}{2} \left(1 \pm \sqrt{1 + 4(-1)^{n+1} \nu_n^3}\right)$$

Now, since $\nu_n = \pi/2 + n\pi$ we have that $4|\nu_n^3| > 1$. Therefore, we have four cases:

(i) If n is odd and $n < 0 \Rightarrow$ fixed point is a stable spiral

(ii) If n is odd and $n > 0 \Rightarrow$ fixed point is a saddle.

(iii) If n is even and $n < 0 \Rightarrow$ fixed point is a saddle.

(iv) If n is even and $n > 0 \Rightarrow$ fixed point is an unstable spiral.

#2

(a) $\dot{x}=x-y$, $\dot{y}=x^2-4$

Solution:

The nullclines are

$\dot{x}=0 \Rightarrow y=x$

$\dot{y}=0 \Rightarrow x=\pm 2$.

Therefore, the fixed points are $(-2,-2)$, $(2,2)$. The Jacobian is:

$$J=\begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$$

$J(-2,-2)$:

$$\begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$$

$\Rightarrow \lambda_1+\lambda_2=1$

$\lambda_1\lambda_2=-4$

$\Rightarrow \lambda_1=1-\lambda_2$

$\Rightarrow \lambda_2(1-\lambda_2)=-4$

$\Rightarrow \lambda_2^2-\lambda_2-4=0$

$\Rightarrow \lambda_2=\frac{1\pm\sqrt{17}}{2}$

$\Rightarrow (-2,-2)$ is a saddle.

$J(2,2)$:

$$\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$$

$\Rightarrow \lambda_1+\lambda_2=1$

$\lambda_1\lambda_2=4$

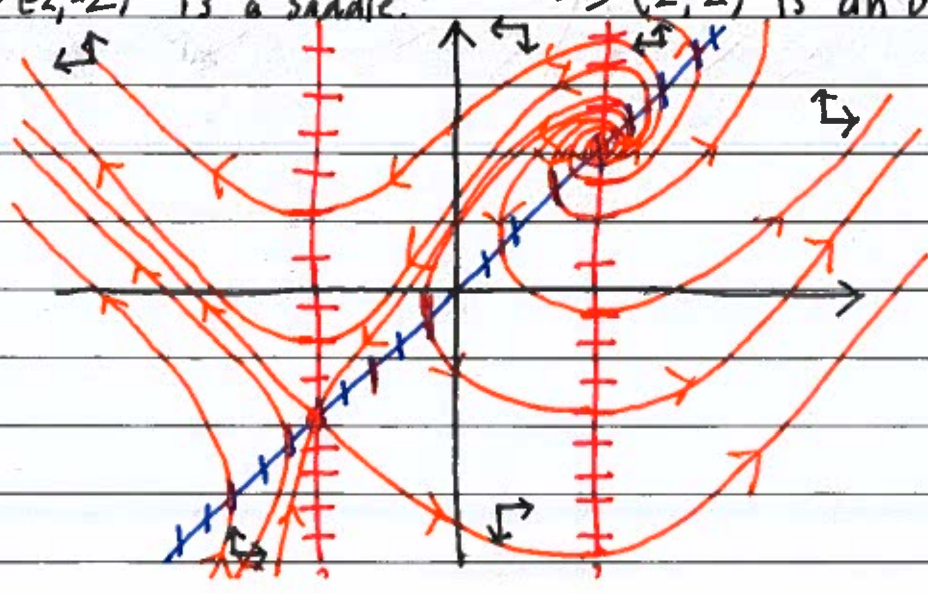
$\Rightarrow \lambda_1=1-\lambda_2$

$\Rightarrow \lambda_1(1-\lambda_1)=4$

$\Rightarrow \lambda_1^2-\lambda_1+4=0$

$\Rightarrow \lambda_1=\frac{1\pm\sqrt{1-16}}{2}$

$\Rightarrow (2,2)$ is an unstable spiral



(b) $\dot{x} = y + x - x^3$

$\dot{y} = -y$

Solution:

Nullclines:

$\dot{x} = 0$: $y = x^3 - x$

$\dot{y} = 0$: $y = 0$

Therefore, the fixed points are $(-1,0), (0,0), (1,0)$. The Jacobian is

$$J = \begin{bmatrix} 1-2x^2 & 1 \\ 0 & -1 \end{bmatrix}$$

$J(0,0)$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow (0,0)$ is a saddle

$J(-1,0)$

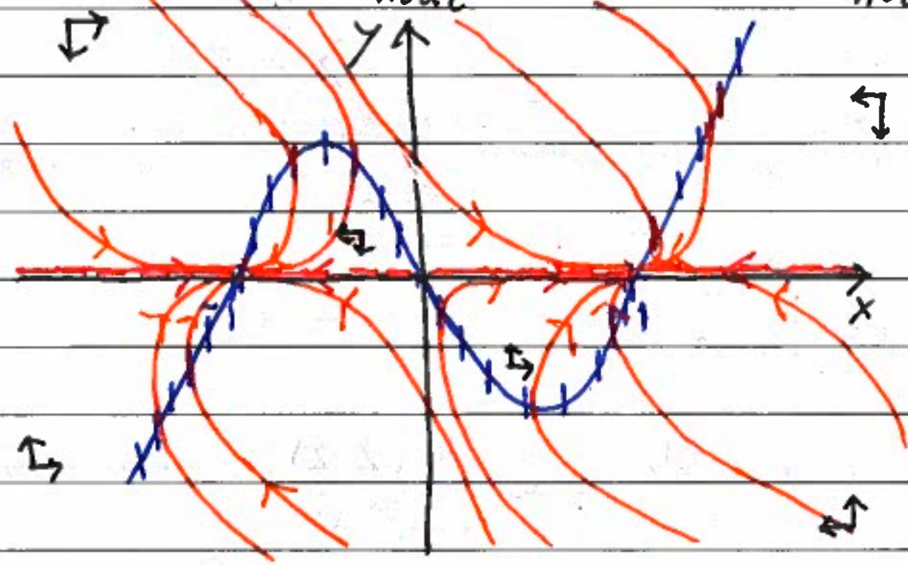
$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

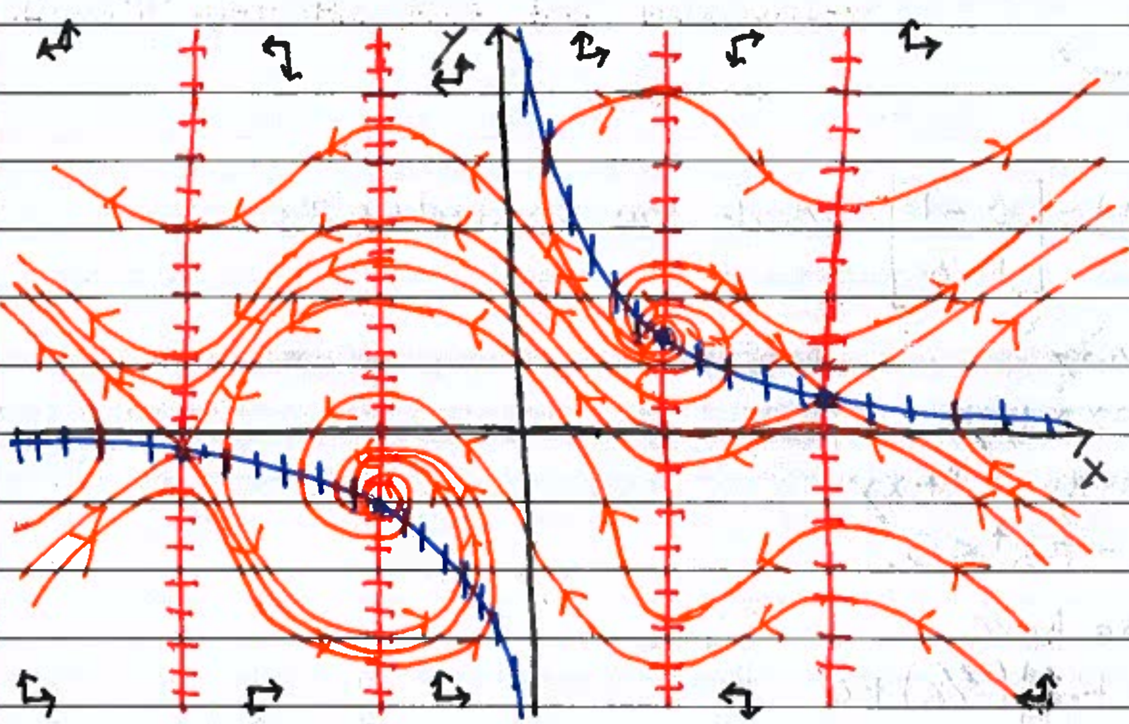
$\Rightarrow (-1,0)$ is a stable node

$J(1,0)$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow (1,0)$ is a stable node.





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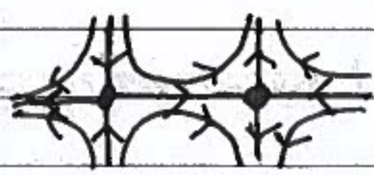
#3

(a)



(b)

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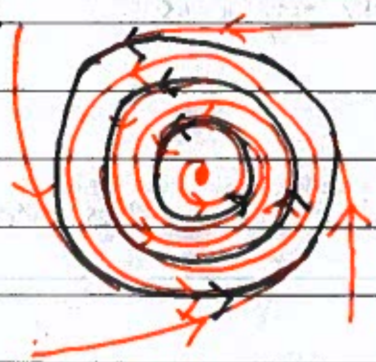
#4



#5

$$\dot{x} = -y - x^3$$
$$\dot{y} = x$$

(a) $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which has eigenvalues $\pm i$.

(b) $r^2 = x^2 + y^2$

$$\Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$$
$$\Rightarrow r\dot{r} = -xy - x^4 + xy$$
$$\Rightarrow \dot{r} = -\frac{1}{r}x^4 \leq 0.$$

We also have

$$\theta = \tan^{-1}(y/x) + C$$
$$\Rightarrow \dot{\theta} = \frac{1}{1+y^2/x^2} \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x^2 - y^2 - yx^3}{r^2} = \frac{r^2\cos^2\theta - r^2\sin^2\theta - r^4\cos^3\theta\sin\theta}{r^4}$$
$$\Rightarrow \dot{\theta} = \frac{\cos 2\theta}{r^2} - \cos^3\theta\sin\theta$$

Since $\dot{r} \leq 0$, the trajectories spiral inward towards the origin.

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#6.

The linearization, i.e, the Jacobian, at the origin is

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which has eigenvalues $\pm i$. Converting to polar coordinates we have

$$r^2 = x^2 + y^2$$
$$\Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$$
$$\Rightarrow r\dot{r} = -xy + ax^4 + yx + ay^4$$
$$\Rightarrow \dot{r} = ax^4 + ay^4$$

Consequently, the origin is an unstable spiral if $a>0$, a stable spiral if $a<0$, and is a center if $a=0$.

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#2

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - \alpha N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - \beta N_1 N_2$$

Solution:

(a) The parameters r_1, r_2 are growth rates, K_1, K_2 are carrying capacities, and α, β are the interaction parameters.

(b) Let $x = N_1/K_1$, $y = N_2/K_2$, and $\tau = r_1 t$. Therefore,

$$\frac{dN_1}{dt} = \frac{d(K_1 x)}{dt} = \frac{d(K_1 x)}{d\tau}\frac{d\tau}{dt} = r_1 K_1 \frac{dx}{d\tau}.$$

Similarily,

$$\frac{dN_2}{dt} = r_2 K_2 \frac{dy}{d\tau}$$

$$\Rightarrow r_1 K_1 \frac{dx}{d\tau} = r_1 K_1 x(1-x) - \alpha K_1 K_2 xy$$
$$r_1 K_2 \frac{dy}{d\tau} = r_2 K_2 y(1-y) - \beta K_1 K_2 xy$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - axy,$$
$$\frac{dy}{d\tau} = by(1-y) - cxy,$$

where

$a = \alpha K_2/r_1$ (dimensionless interaction parameter)

$b = r_2/r_1$ (dimensionless ratio of growth rate)

$c = \beta K_1/r_1$ (dimensionless interaction parameter)

(c) Without loss of generality, we can assume $b \geq 1$ since if $b < 1$ we could just relabel x and y to ensure $r_2 > r_1$. The nullclines for this system are:

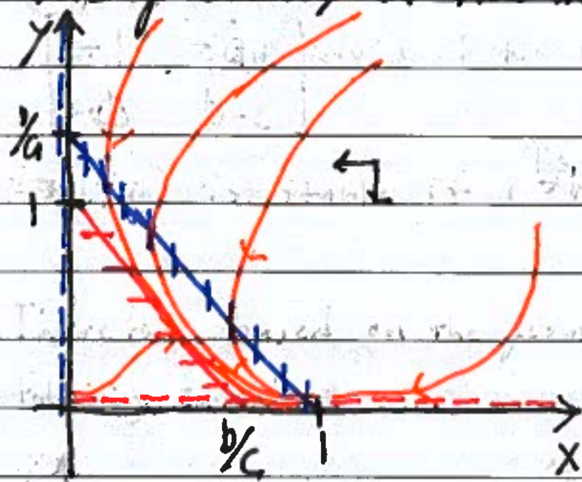
$x = 0$ ($\dot{x} = 0$)

$y = -\frac{1}{a}(x-1)$ ($\dot{x} = 0$)

$y = 0$ ($\dot{y} = 0$)

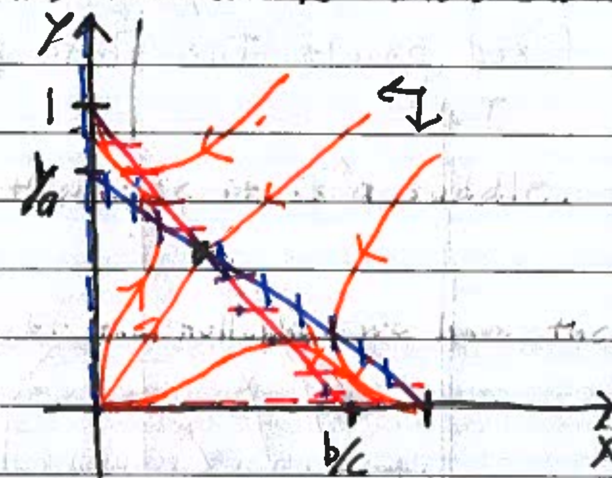
$x = -\frac{b}{c}(y-1)$ ($\dot{y} = 0$)

The geometry of the nullclines dictates the various cases.



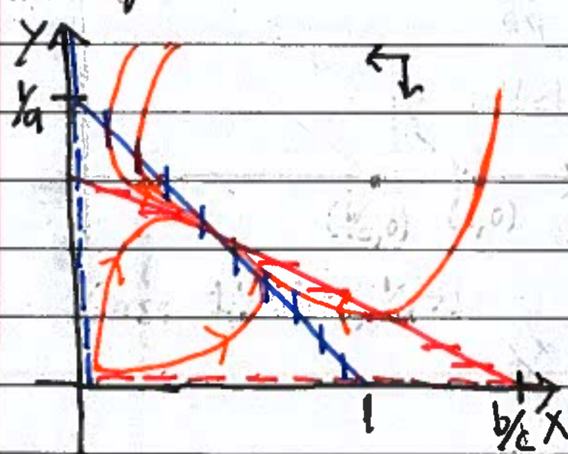
$b/c < 1$, $1/a \geq 1$

(y goes extinct)



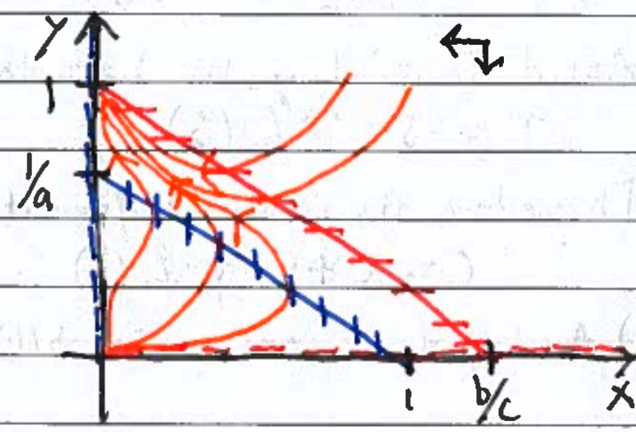
$b/c < 1$, $1/a < 1$

(x or y goes extinct)



$b/c > 1$, $1/a > 1$

(x and y coexist)



$b/c > 1$, $1/a < 1$

(x goes extinct)

#10

$\dot{S} = -\beta S I$

$\dot{I} = \beta S I - \alpha I.$

Solution:

(a) The nullclines are

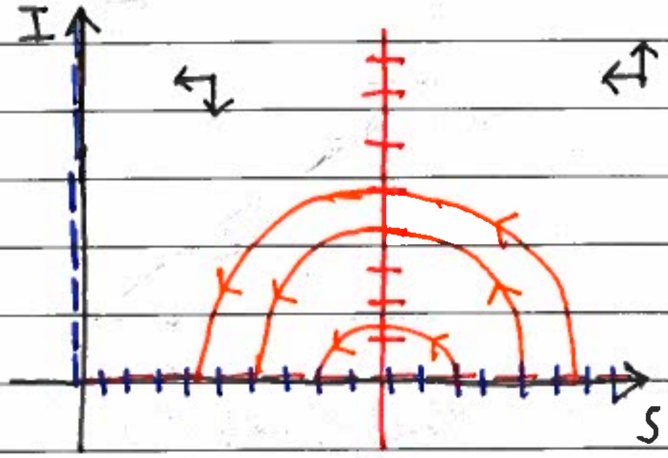
$S = 0, \dot{S} = 0$

$I = 0, \dot{S} = 0$

$I = 0, \dot{I} = 0$

$S = \alpha/\beta, \dot{I} = 0$

Since $I=0$ is a repeated nullcline it follows that it is a line of fixed points. The phase portrait is given by:



(b) $\dfrac{dI}{dS} = \dfrac{\beta S I - \alpha I}{-\beta S I} = -\dfrac{\beta S - \alpha}{\beta S} = -1 - \dfrac{\alpha}{\beta S}$

Antidifferentiating, we have the following

$$I = -S - \frac{\alpha}{\beta} \ln(S) + C$$

Therefore, the conserved quantity is

$$C = I + S - \frac{\alpha}{\beta} \ln(S).$$

(c) An epidemic occurs if $S(0) = \alpha/\beta$. ■

#11

$\dot{P} = P(R-S) = PR - PS$

$\dot{R} = R(S-P) = RS - RP$

$\dot{S} = S(P-R) = SP - SR$

Solution:

(a) Each of these equations models the effect that when paper encounters rock, paper wins and prospers and rock dies and so on for the encounters of scissors and rock and scissors and paper.

(b-c) If we let $N = P+R+S$ it follows that $\dot{N} = \dot{P} + \dot{R} + \dot{S} = 0$. We also have that if we let $E = PRS$ then

$$\dot{E} = \dot{P}RS + P\dot{R}S + PR\dot{S} = P(R-S)RS + PR(S-P)S + PRS(P-R)$$

$\Rightarrow \dot{E} = PRS(R-S+S-P+P-R) = 0.$

(d) If we let

$$x = P/N,\ y = R/N,\ z = R/N,\ \tau = Nt$$

we have the equivalent system

$$\frac{dx}{d\tau} = x(y-z)$$
$$\frac{dy}{d\tau} = y(z-x)$$
$$\frac{dz}{d\tau} = z(x-y)$$

where $x+y+z=1$. Removing the z-equation we have:

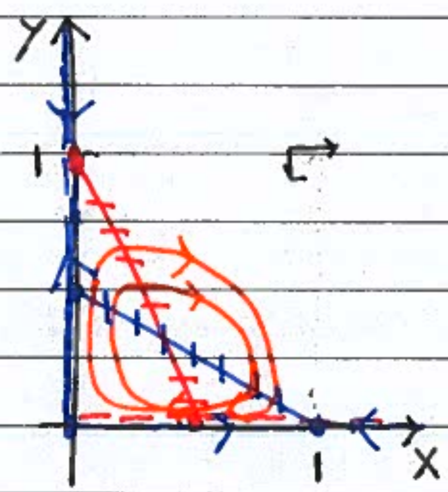
$$\frac{dx}{d\tau} = x(y-1+x+y) = x(2y-1+x)$$

$$\frac{dy}{d\tau} = y(1-x-y-x) = y(1-2x-y)$$

which is a two-dimensional system with the conserved quantity

$$E = xy(1-x-y).$$

The phase portrait cannot have any attractors or repellors and thus we can analyze the null-clines



Therefore, unless one or more species are extinct, i.e. $x=0, y=0$, or $z=0$, the system predicts that the species population oscillate in time.

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