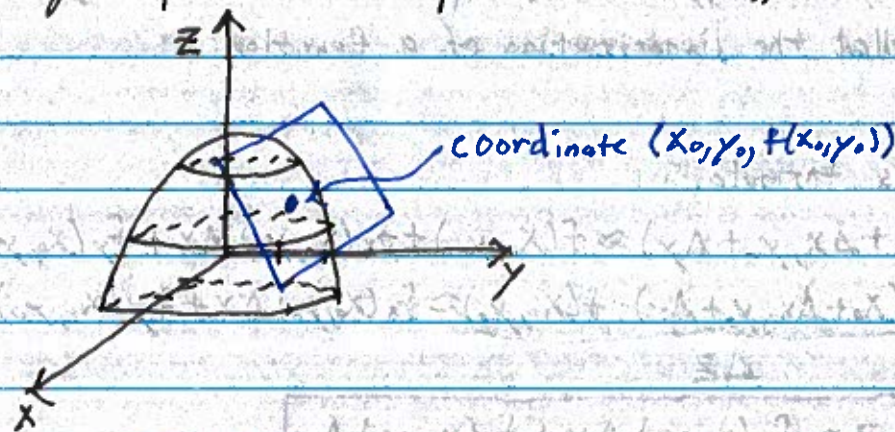


Lecture #10: Tangent Planes, Linear Approximation, Chain Rule

Suppose we have a surface $z = f(x, y)$ as sketched below with its tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.



How can we find the tangent plane $L(z)$?

$$\Rightarrow L(x, y) = a + b(x - x_0) + c(y - y_0)$$

• Contact condition: $L(x_0, y_0) = f(x_0, y_0)$

$$\Rightarrow a = f(x_0, y_0)$$

• Slope in x-direction: $L_x(x_0, y_0) = f_x(x_0, y_0)$

$$\Rightarrow b = f_x(x_0, y_0)$$

• Slope in y-direction: $L_y(x_0, y_0) = f_y(x_0, y_0)$

$$\Rightarrow c = f_y(x_0, y_0)$$

Therefore, the equation of the tangent plane is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Contact
point

Slope in
x-direction

Slope in
y-direction

Linear Approximations

Near a point (x_0, y_0) the tangent plane can be used to approximate a function:

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

this is called the linearization of a function.

Using this formula,

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

$$\Rightarrow \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta z} \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

$$\Rightarrow \boxed{\Delta z \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}$$

(Approximate Change of Function for small change Δx and Δy)

Example:

Two rollers are used to make copper wire; see figure below. If the rollers spin at a rate of 10 m/s and the temperature of the metal is 900°C the produced wire has a gauge of 4mm. Also, if the speed is increased by .2 m/s the gauge increases by .06mm while if the temperature increases by 10° the gauge decreases by .04mm. Estimate the gauge produced at 10.1 m/s and 880°C .

Assume gauge is a function of speed and temperature; i.e. $g(S, T)$.

$$1. g(10, 900) = 4$$

$$4. \Delta S = 10.1 - 10 = .1$$

$$2. \frac{\partial g}{\partial S} = \frac{.06}{.2} = .3$$

$$5. \Delta T = 880 - 900 = -20$$

$$3. \frac{\partial g}{\partial T} = \frac{-.04}{10} = -.004$$

Therefore,

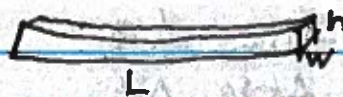
$$\Delta g = .3(.1) + .004(-20) = .03 + .08 = .11$$

$$\Rightarrow g(10.1, 880) \approx 4 + .11 = 4.11\text{mm}$$

Example:

The sag in a beam of length L , width W , and height h is given by

$$S(L, W, h) = \frac{cL^4}{wh^3}$$



Which variable has the greatest effect on sag?

$$\frac{\partial S}{\partial L} = \frac{4cL^3}{wh^3} = \frac{4cL^4}{Lwh^3} = 4 \frac{S}{L} \Rightarrow \Delta S = 4 \frac{\Delta L}{L} S$$

(Change in S is 4 times change in length)

$$\frac{\partial S}{\partial W} = -\frac{cL^4}{w^2h^3} = -\frac{S}{W} \Rightarrow \Delta S = -\frac{\Delta W}{W} S$$

(Changing W produces a decrease in S that is same as change in width)

$$\frac{\partial S}{\partial h} = -\frac{3cL^4}{wh^4} = -3 \frac{S}{h} \Rightarrow \Delta S = -3 \frac{\Delta h}{h} S$$

(Changing h produces a decrease in S that is 3 times the change in height)

Chain Rule:

$$\begin{aligned}\frac{d}{dt}(f(x(t), y(t))) &= \lim_{\Delta t \rightarrow 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left(f_x(x(t), y(t)) \frac{\Delta x}{\Delta t} + f_y(x(t), y(t)) \frac{\Delta y}{\Delta t} \right) \\ &= f_x(x, y) \frac{dx}{dt} + f_y(x, y) \frac{dy}{dt}\end{aligned}$$

$$\Rightarrow \boxed{\frac{d}{dt}(f(x(t), y(t))) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}} \rightarrow \text{Chain Rule.}$$

Example:

$$f(x, y) = x^2 e^y, \quad x(t) = t^2 - 1, \quad y(t) = \sin(t)$$

• Direct approach:

$$f(x(t), y(t)) = (t^2 - 1)^2 e^{\sin(t)}$$

$$\Rightarrow \frac{df}{dt} = 2(t^2 - 1)e^{\sin(t)} \cdot 2t + (t^2 - 1)^2 \cos(t) e^{\sin(t)}$$

• New approach

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$\Rightarrow \frac{df}{dt} = 2x e^y \cdot 2t + x^2 e^y \cos(t)$$

$$= 2(t^2 - 1)e^{\sin(t)} \cdot 2t + (t^2 - 1)^2 e^{\sin(t)} \cos(t)$$

Theorem - If $z = f(x(s, t), y(s, t))$ then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

Example:

Suppose $z = f(x, y)$. What is $\frac{\partial z}{\partial r}$ in polar coordinates?

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

We know

$$x = r \cos \theta \text{ and } y = r \sin \theta \\ \Rightarrow \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta$$

Therefore,

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial f}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial f}{\partial y}$$

What is $\frac{\partial z}{\partial \theta}$ in polar coordinates?

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

$$\Rightarrow \frac{\partial z}{\partial \theta} = -r \sin \theta \frac{\partial f}{\partial x} + r \cos \theta \frac{\partial f}{\partial y} = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y}$$

We also have

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x}$$

$$r^2 = x^2 + y^2$$

$$\Rightarrow 2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta$$

$$\tan \theta = \frac{y}{x}$$

$$\Rightarrow \sec^2 \theta \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = -\cos^2 \theta \cdot \frac{r \sin \theta}{r^2 \cos^2 \theta} = -\frac{1}{r} \sin \theta$$

Therefore,

$$\frac{\partial z}{\partial x} = \cos \theta \frac{\partial f}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial f}{\partial \theta}$$

We also have:

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial y}$$

$$r^2 = x^2 + y^2$$
$$\Rightarrow 2r \frac{\partial r}{\partial y} = 2y$$

$$\Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta$$

$$\tan \theta = \frac{y}{x}$$
$$\Rightarrow \sec^2 \theta \frac{\partial \theta}{\partial y} = \frac{1}{x}$$

$$\Rightarrow \frac{\partial \theta}{\partial y} = \frac{\cos^2 \theta}{r \cos \theta} = \frac{\cos \theta}{r}$$

Therefore,

$$\frac{\partial z}{\partial y} = \sin \theta \frac{\partial f}{\partial r} + \cos \theta \frac{\partial f}{\partial \theta}$$