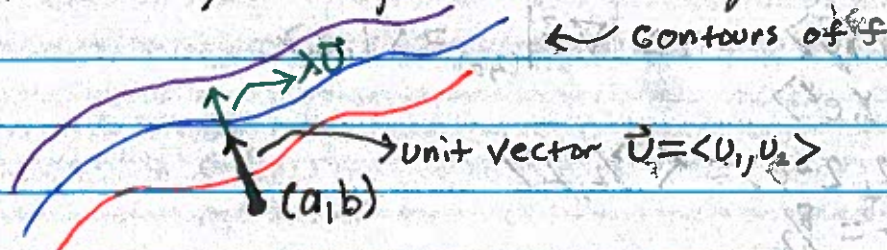


Lecture #11: Directional Derivatives and Gradients

Directional Derivatives:

How do you compute rate of change in different directions?



$$D_u f(a, b) = \lim_{\lambda \rightarrow 0} \frac{f(a + \lambda u_1, b + \lambda u_2) - f(a, b)}{\lambda}$$

If we let $g(\lambda) = f(a + \lambda u_1, b + \lambda u_2)$, $x = a + \lambda u_1$, $y = b + \lambda u_2$, then $D_u f(a, b) = \left. \frac{dg}{d\lambda} \right|_{\lambda=0}$

But

$$\begin{aligned} \frac{dg}{d\lambda} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \lambda} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \lambda} \\ &= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 \end{aligned}$$

$$\Rightarrow \left. \frac{dg}{d\lambda} \right|_{\lambda=0} = D_u f(a, b) = f_x(a, b) u_1 + f_y(a, b) u_2$$

New Symbol:

$$\begin{aligned} D_u f(a, b) &= f_x(a, b) u_1 + f_y(a, b) u_2 \\ &= \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle \\ &= \underbrace{\nabla f}_{\text{gradient of } f} \cdot \vec{u} \end{aligned}$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Example:

$$f(x, y) = xe^y$$

1. Find the rate of change of f at the point $P=(2, 0)$ in the direction from P to $Q=(\frac{1}{2}, 2)$.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \Rightarrow \nabla f|_{(2,0)} = \langle 1, 2 \rangle$$
$$= \langle e^y, xe^y \rangle$$

$$\vec{U} = \langle \frac{1}{2} - 2, 2 - 0 \rangle = \langle -\frac{3}{2}, 2 \rangle$$

$$\|\vec{U}\| = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$$

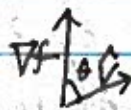
The unit vector is

$$\hat{U} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$$

$$\Rightarrow D_{\hat{U}} f = \nabla f|_{(2,0)} \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = \langle 1, 2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = -\frac{3}{5} + \frac{8}{5} = 1.$$

2. What is the direction in which the function grows the fastest? For any direction $\hat{V}$ we know that

$$D_{\hat{V}} f = \nabla f \cdot \hat{V}$$
$$= \|\nabla f\| \cdot \|\hat{V}\| \cos \theta$$
$$= \|\nabla f\| \cdot \cos \theta$$



This is largest when $\theta=0$, i.e. $\hat{V}$ is parallel with ∇f .

Big Result:

- ∇f points in the direction of greatest increase.
- $-\nabla f$ points in the direction of greatest decrease.
- $\|\nabla f\|$ measures the rate of greatest increase.

In this case:

$$\|\nabla f|_{(2,0)}\| = \|\langle 1, 2 \rangle\| = \sqrt{5}.$$