

Lecture #12: Optimization

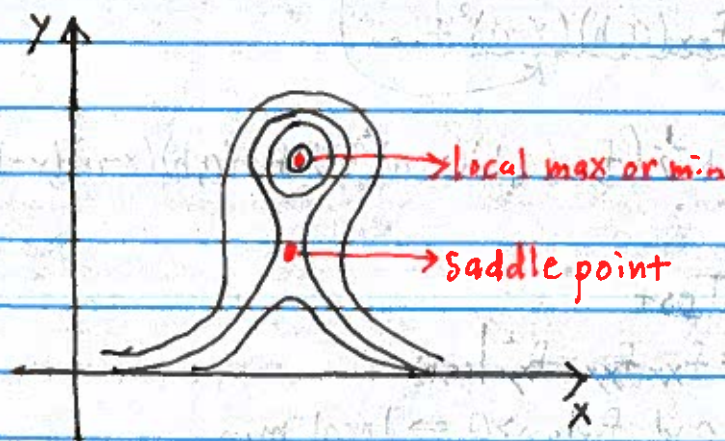
Extreme Values:

1. $f(a,b)$ is a local maximum of f if $f(a,b) > f(x,y)$ for all (x,y) in a neighborhood of (a,b) .
2. $f(a,b)$ is a local minimum of f if $f(a,b) < f(x,y)$ for all (x,y) in a neighborhood of (a,b) .

Theorem- If $f(x,y)$ has a local max or min at a point (a,b) and if its partial derivatives exist, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

* A point where $f_x = 0$ and $f_y = 0$ is called a critical point.

Definition- A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if there exists directions $\vec{u}, \vec{v}$ for which $D_{\vec{u}}f > 0$ and $D_{\vec{v}}f < 0$.



Critical Points of Quadratics

$$f(x,y) = ax^2 + bxy + cy^2$$

$$\nabla f = \langle 2ax + by, bx + 2cy \rangle$$

Critical point at $(0,0)$.

$$f(x,y) = a(x^2 + \frac{b}{a}xy + \frac{c}{a}y^2)$$

$$= a(x^2 + \frac{b}{a}xy + \frac{b^2}{4a^2}y^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2)$$

$$= a(x + \frac{b}{2a}y)^2 + a(\frac{c}{a} - \frac{b^2}{4a^2})y^2$$

$$= a((x + \frac{b}{2a}y)^2 + (\frac{c}{a} - \frac{b^2}{4a^2})y^2)$$

- Local max if: $a < 0$, $4ac - b^2 > 0$
- Local min if: $a > 0$, $4ac - b^2 > 0$
- Saddle point if: $4ac - b^2 < 0$

Taylor's Theorem

Suppose (a,b) is a critical point

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \dots$$

$$= f(a,b) + f_x(a,b)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2 + f_x(a,b)(x-a) + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \dots$$

$$\Rightarrow f(x,y) \approx f(a,b) + \frac{1}{2}(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2)$$

Second Derivative Test

$$\text{Define } D(a,b) = f_{xx}f_{yy} - f_{xy}^2|_{(a,b)}.$$

(i) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$ local min

(ii) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$ local max

(iii) If $D(a,b) < 0$ then f is a saddle

(iv) If $D(a,b) = 0$ you know nothing

Example

$$f(x,y) = 2x^2 - y^3 - 2xy$$

$$\Rightarrow \nabla f = \langle 4x - 2y, -3y^2 - 2x \rangle$$

Critical points satisfy:

$$4x - 2y = 0, -3y^2 - 2x = 0$$

$$\Rightarrow y = 2x, -12x^2 - 2x = 0$$

$$\Rightarrow y = 2x, -2x(6x + 1) = 0$$

$$\Rightarrow y = 2x, x = 0, x = -\frac{1}{6}$$

The critical points are therefore:

$$(0,0) \text{ and } (-\frac{1}{6}, -\frac{1}{3})$$

Now

$$f_{xx} = 4, f_{xy} = -2, f_{yx} = -6y$$

$(0,0)$:

$$f_{xx}|_{(0,0)} = 4, f_{xy}|_{(0,0)} = -2, f_{yx}|_{(0,0)} = 0$$

$$\Rightarrow D(0,0) = 4 \cdot 0 - (-2)^2 = -4$$

Saddle point

$(-\frac{1}{6}, -\frac{1}{3})$:

$$f_{xx}|_{(-\frac{1}{6}, -\frac{1}{3})} = 4, f_{xy}|_{(-\frac{1}{6}, -\frac{1}{3})} = -2, f_{yx}|_{(-\frac{1}{6}, -\frac{1}{3})} = 2$$

$$\Rightarrow D(-\frac{1}{6}, -\frac{1}{3}) = 4 \cdot 2 - (-2)^2 = 8 - 4 = 4$$

Local Minimum

Example

$$f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2y$$

$$\nabla f = \langle 3x^2 + 6xy, -4y - 8y^3 + 3x^2 \rangle$$

Critical points satisfy:

$$3x^2 + 6xy = 0, -4y - 8y^3 + 3x^2 = 0$$

$$\Rightarrow x(3x + 6y) = 0, -4y - 8y^3 + 3x^2 = 0$$

$$\Rightarrow x = 0 \text{ or } x = -2y, -4y - 8y^3 = 0 \text{ or } -4y - 8y^3 + 12y^2 = 0$$

$$\Rightarrow x = 0 \text{ or } x = -2y, -4y(1 + 2y^2) = 0 \text{ or } -4y(2y^2 - 3y + 1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = -2y, -4y(1 + 2y^2) = 0 \text{ or } -4y(2y - 1)(y - 1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = -2y, y = 0 \text{ or } y = \frac{1}{2}, y = 1$$

The critical points are therefore

$$(0,0), (-1, \frac{1}{2}), (-2,1)$$

$$f_{xx} = 6x + 6y, f_{yy} = -4 - 24y^2, f_{xy} = 6x$$

• $D(0,0)$ Indeterminate

• $D(-2,1) = (-12 + 6)(-28) - 144 = 24$ (bowl)

$f_{xx}(-2,1) < 0$ local maximum.

• $D(-1, \frac{1}{2}) = -6$ (saddle)

Extreme Value Theorem - Suppose f is continuous on a closed and bounded region $R \subset \mathbb{R}^2$. Then f has an absolute maximum and minimum in R . Furthermore, max/min occurs at either critical points or the boundary.

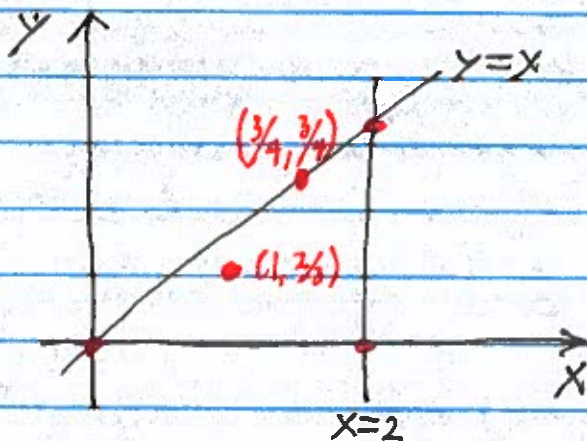
Example:

Find the absolute extrema of $f(x,y) = x^2 + 3y - 3xy$ on the region R bounded by the lines

$$y=x$$

$$y=0$$

$$x=2$$



$$\nabla f = \langle 2x - 3y, 3 - 3x \rangle$$

$$\nabla f = 0 \Rightarrow 2x - 3y = 0 \text{ and } 3 - 3x = 0$$

$$\Rightarrow x=1 \text{ and } y=\frac{2}{3}$$

$$f(1, \frac{2}{3}) = 1 + 2 - 2$$

$$\Rightarrow \boxed{f(1, \frac{2}{3}) = 1}$$

Test Along Lines

1. If $y=x$, $f(x,x) = x^2 + 3x - 3x^2$ and $x \in [0,2]$

$$\Rightarrow \frac{d}{dx} f(x,x) = 2x + 3 - 6x \\ = -4x + 3$$

Critical point at $x=\frac{3}{4}$, $y=\frac{3}{4}$

$$f(0,0)=0, f(2,2)=-2, f(\frac{3}{4}, \frac{3}{4}) = \frac{9}{16} + \frac{9}{4} - 2\frac{9}{16}$$

$$\Rightarrow \boxed{f(0,0)=0}, \boxed{f(2,2)=-2}, \boxed{f(\frac{3}{4}, \frac{3}{4})=0}$$

2. If $y=0$, $f(x,0) = x^2$

Critical point at $x=0$.

$$\boxed{f(0,0)=0}, \boxed{f(2,0)=4}$$

3. If $x=2$, $f(2,y) = 4 - 3y$

No critical points and already test boundary points.

$\Rightarrow$ Max is 4 at $(2,0)$ and min is -2 at $(2,2)$